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Part I

Elementary electromagnetic waves
Chapter 1

Maxwell’s equations, the material equations, and boundary conditions

In this course we consider light to be electromagnetic waves of frequencies $\nu$ in the visible range, so that $\nu \simeq (4 - 7.5) \times 10^{14}$ Hz. Since $\lambda = \frac{c}{\nu}$, where $c$ is the speed of light in vacuum ($c \simeq 3 \times 10^8$ m/s), we find that the corresponding wavelength interval is $\lambda \simeq (0.4 - 0.75) \mu m$. Thus, to study the propagation of light we must consider the propagation of the electromagnetic field, which is represented by the two vectors $E$ and $B$, where $E$ is the electric field strength and $B$ is the magnetic induction or the magnetic flux density. To enable us to describe the interaction of the electromagnetic field with material objects we need three additional vector quantities, namely the current density $J$, the displacement $D$, and the magnetic field strength $H$.

1.1 Maxwell’s equations

The five vectors mentioned above are linked together by Maxwell’s equations, which in Gaussian units are

$$\nabla \times H = \frac{1}{c} \dot{D} + \frac{4\pi}{c} J, \quad (1.1.1)$$

$$\nabla \times E = -\frac{1}{c} \dot{B}. \quad (1.1.2)$$

In addition we have the two scalar equations

$$\nabla \cdot D = 4\pi \rho, \quad (1.1.3)$$

$$\nabla \cdot B = 0, \quad (1.1.4)$$

where $\rho$ is the charge density. Equation (1.1.3) can be said to define the charge density $\rho$. Similarly, we can say that (1.1.4) implies that free magnetic charges do not exist.

1.2 The continuity equation

The charge density $\rho$ and the current density $J$ are not independent quantities. By taking the divergence of (1.1.1) and using that $\nabla \cdot (\nabla \times A) = 0$ for an arbitrary vector $A$, we find that

$$\nabla \cdot J + \frac{1}{4\pi} \nabla \cdot \dot{D} = 0,$$
which on using (1.1.3) gives
\[ \nabla \cdot \mathbf{J} + \dot{\rho} = 0. \] (1.2.1)
This equation is called the continuity equation, and it expresses conservation of charge. By integrating (1.2.1) over a closed volume \( V \) with surface \( S \), we find
\[ \iiint_V \nabla \cdot \mathbf{J} \, dv = -\iiint_V \frac{\partial \rho}{\partial t} \, dv, \] (1.2.2)
which by use of the divergence theorem gives
\[ \oint_S \mathbf{J} \cdot \hat{n} \, da = -\frac{d}{dt} \iiint_V \rho \, dv = -\frac{d}{dt} Q. \] (1.2.3)
Here \( \hat{n} \) is the unit surface normal in the direction out of the volume \( V \), so that (1.2.3) shows that the integrated current flux out of the closed volume \( V \) is equal to the loss of charge in the same volume.

Digression 1: Notation
- **Bold face** is used to denote vector quantities, e.g.
  \[ \mathbf{E} = E_x \hat{e}_x + E_y \hat{e}_y + E_z \hat{e}_z, \]
  where \( \hat{e}_x, \hat{e}_y, \) and \( \hat{e}_z \) are unit vectors along the axes in a Cartesian co-ordinate system.
- A dot above a symbol is used to denote the time derivative, e.g.
  \[ \dot{\mathbf{B}} = \frac{\partial}{\partial t} \mathbf{B}. \]
- \( \mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}, \rho, \) and \( \mathbf{J} \) are functions of the position \( \mathbf{r} \) and the time \( t \), e.g.
  \[ \mathbf{D} = \mathbf{D}(\mathbf{r}, t). \]
- The connection between Gaussian and other systems of units, e.g. MKS units, follows from J.D. Jackson, "Classical Electrodynamics", Wiley (1962), pp. 611-621. For conversion between Gaussian units and MKS units, we refer to the table on p. 621 in this book.

1.3 The material equations
Maxwell’s equations (1.1.1)-(1.1.4), which connect the fundamental quantities \( \mathbf{E}, \mathbf{H}, \mathbf{B}, \mathbf{D}, \) and \( \mathbf{J} \), are not sufficient to uniquely determine the field vectors (\( \mathbf{E}, \mathbf{B} \)) from a given distribution of currents and charges. In addition we need the so-called material equations, which describe how the field is influenced by matter.

In general the material equations can be relatively complicated. But if the field is time harmonic and the matter is isotropic and at rest, the material equations have the following simple form
\[ \mathbf{J}_c = \sigma \mathbf{E}, \] (1.3.1)
\[ \mathbf{D} = \varepsilon \mathbf{E}, \] (1.3.2)
\[ \mathbf{B} = \mu \mathbf{H}, \]  
\[ (1.3.3) \]

where \( \sigma \) is the conductivity, \( \varepsilon \) is the permittivity or dielectric constant, and \( \mu \) is the permeability.

Equation (1.3.1) is Ohm’s law, and \( \mathbf{J}_c \) is the conduction current density, which arises because the material has a non-vanishing conductivity \( (\sigma \neq 0) \). The total current density \( \mathbf{J} \) in (1.1.1) can in addition consist of an externally applied current density \( \mathbf{J}_0 \), so that

\[ \mathbf{J} = \mathbf{J}_0 + \mathbf{J}_c = \mathbf{J}_0 + \sigma \mathbf{E}. \]  
\[ (1.3.4) \]

**Digression 2: General material considerations**

- A material that has a non-negligible conductivity \( \sigma \) is called a conductor, while a material that has a negligible conductivity is called an insulator or a *dielectric*.
- Metals are good conductors.
- Glass is a dielectric; \( \varepsilon \approx 2.25; \sigma = 0; \mu = 1 \).
- In anisotropic media (e.g. crystals) the relation in (1.3.2) is to be replaced by \( \mathbf{D} = \xi \mathbf{E} \), where \( \xi \) is a tensor, dyadic or matrix.
- In a plasma (1.3.1) is to be replaced by \( \mathbf{J} = \sigma \mathbf{E} \), where the conductivity is a tensor. Thus, in this case the permeability is a tensor. Such materials are not important in optics.
- In dispersive media \( \varepsilon \) is frequency dependent, i.e. \( \varepsilon = \varepsilon(\omega) \). Maxwell’s equations and the material equations are still valid for each frequency component or time harmonic component of the field. For a pulse consisting of many frequency components, one must apply Fourier analysis to solve Maxwell’s equations and the material equations separately for each time harmonic component, and then perform an inverse Fourier transformation.
- In *non-linear* media there is no linear relation between \( \mathbf{D} \) and \( \mathbf{E} \) (equation (1.3.2) is not valid). Most media become non-linear when the electric field strength becomes sufficiently high.

### 1.4 Boundary conditions

Hitherto we have assumed that \( \varepsilon \) and \( \mu \) are continuous functions of the position. But in optics we often have systems consisting of several different types of glass. At the transition between air and glass or between two different types of glass the material parameters are discontinuous. Let us therefore consider what happens to the electromagnetic field at the boundary between two media.

Consider two media that are separated by an interface, as illustrated in Fig. 1.1. From Maxwell’s equations, combined with Stokes’ and Gauss’ theorems, one can derive the following boundary conditions

\[ \mathbf{n} \cdot (\mathbf{B}^{(2)} - \mathbf{B}^{(1)}) = 0, \]  
\[ (1.4.1) \]

\[ \mathbf{n} \cdot (\mathbf{D}^{(2)} - \mathbf{D}^{(1)}) = 4\pi \rho_s, \]  
\[ (1.4.2) \]

\[ \mathbf{n} \times (\mathbf{E}^{(2)} - \mathbf{E}^{(1)}) = 0, \]  
\[ (1.4.3) \]

\[ \mathbf{n} \times (\mathbf{H}^{(2)} - \mathbf{H}^{(1)}) = \frac{4\pi}{c} \mathbf{J}_s, \]  
\[ (1.4.4) \]

where \( \mathbf{n} \) is a unit vector along the surface normal. According to (1.4.1) the normal component of \( \mathbf{B} \) is continuous across the boundary, while (1.4.2) says that if there exists a surface charge density
Figure 1.1: A plane interface with unit normal $\hat{n}$ separates two different dielectric media.

$\rho_s$ at the boundary, then the normal component of $D$ is changed by $4\pi\rho_s$ across the boundary between the two media. According to (1.4.3) the tangential component of $E$ is continuous across the boundary, while (1.4.4) implies that if there exists a surface current density $J_s$ at the boundary, then the tangential component of $H$, i.e. of $\hat{n} \times H$, is changed by $\frac{4\pi}{\varepsilon} J_s$. 
Chapter 2

Poynting’s vector and the energy law

The electric energy density \(w_e\) and the magnetic energy density \(w_m\) are defined by

\[
w_e = \frac{1}{8\pi} \mathbf{E} \cdot \mathbf{D}, \tag{2.1}
\]

\[
w_m = \frac{1}{8\pi} \mathbf{H} \cdot \mathbf{B}, \tag{2.2}
\]

and the total energy density is the sum of these, i.e.

\[
w = w_e + w_m. \tag{2.3}
\]

The energy flux of the field is represented by Poynting’s vector \(\mathbf{S}\), given by

\[
\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}. \tag{2.4}
\]

Here \(\mathbf{S}\) represents the amount of energy that per unit time crosses a unit area that is parallel with both \(\mathbf{E}\) and \(\mathbf{H}\).

In a non-conducting medium (\(\sigma = 0\)) we have the conservation law

\[
\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} = 0, \tag{2.5}
\]

which expresses that the change of the energy density in a small volume is equal to the energy flux out of the same volume [cf. (1.2.2) and (1.2.3)]. In optics the Poynting vector is very important, because its absolute value is proportional to the light intensity, i.e.

\[
|\mathbf{S}| \propto \text{light intensity}. \tag{2.6}
\]

The direction of the Poynting vector, defined by the unit vector

\[
\hat{s} = \frac{\mathbf{S}}{|\mathbf{S}|}, \tag{2.7}
\]

points in the direction of light propagation.
Chapter 3

The wave equation and the speed of light

The electric and magnetic fields $E$ and $H$ are connected through Maxwell’s equations (1.1.1)-(1.1.4), which are simultaneous, first-order partial differential equations. But in those parts of space where there are no sources, so that $J = 0$ and $\rho = 0$, we can through differentiation obtain second-order partial differential equations that $E$ and $H$ satisfy individually. We assume that the medium is non-dispersive, so that $D = \varepsilon E$, where $\dot{\varepsilon} = 0$, and $B = \mu H$, where $\dot{\mu} = 0$. Then we have from (1.1.1) and (1.1.2)

$$\nabla \times H = \frac{1}{c} \dot{D} = \frac{1}{c} \varepsilon \dot{E}, \quad (3.1)$$

$$\nabla \times E = -\frac{1}{c} \dot{B} = -\frac{1}{c} \mu \dot{H}, \quad (3.2)$$

Next, we assume that the medium is homogeneous, so that $\varepsilon$ and $\mu$ do not vary with position. By taking the curl of (3.2) and combining the result with the time derivative of (3.1), we find that

$$\nabla \times (\nabla \times E) = -\frac{1}{c} \mu \nabla \times \dot{H} = -\frac{1}{c} \mu \frac{1}{c} \varepsilon \dot{E} = -\frac{\varepsilon \mu}{c^2} \ddot{E}. \quad (3.3)$$

Now we use the vector relation

$$\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A, \quad (3.4)$$

which applies to an arbitrary vector $A$, to obtain

$$\nabla (\nabla \cdot E) - \nabla^2 E = -\frac{\varepsilon \mu}{c^2} \ddot{E}, \quad (3.5)$$

which since $\nabla \cdot E = 0$, gives

$$\nabla^2 E - \frac{\varepsilon \mu}{c^2} \ddot{E} = 0. \quad (3.6)$$

In a similar manner we find

$$\nabla^2 H - \frac{\varepsilon \mu}{c^2} \ddot{H} = 0. \quad (3.7)$$

By comparing these results with the scalar wave equation

$$\nabla^2 V - \frac{1}{v^2} \ddot{V} = 0, \quad (3.8)$$

we see that in a source-free region of space each Cartesian component of $E$ and $H$ satisfies the scalar wave equation with phase velocity
\[ v = \frac{c}{\sqrt{\varepsilon \mu}}. \]  \hspace{1cm} (3.9)

Note that this derivation is valid only in a non-dispersive medium in which both the permittivity and the permeability do not depend on the frequency.
Chapter 4

Scalar waves

Scalar waves are solutions of the scalar wave equation (3.8), which is given by

$$\nabla^2 V(r,t) - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} V(r,t) = 0. \quad (4.0.1)$$

4.1 Plane waves

Any solution of (4.0.1) of the form

$$V(r,t) = V(r \cdot \hat{s}, t), \quad (4.1.1)$$

is called a plane wave, since $V$ at any time $t$ is constant over any plane

$$r \cdot \hat{s} = \text{constant}, \quad (4.1.2)$$

which is normal to the unit vector $\hat{s}$ (see Fig. 4.1).

To show that (4.1.1) is a solution of (4.0.1), we introduce a new variable

$$\zeta = r \cdot \hat{s} = xs_x + ys_y + zs_z, \quad (4.1.3)$$

so that

$$\frac{\partial \zeta}{\partial x} = s_x; \quad \frac{\partial \zeta}{\partial y} = s_y; \quad \frac{\partial \zeta}{\partial z} = s_z. \quad (4.1.4)$$

Further we find that

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x} = s_x \frac{\partial V}{\partial \zeta}. \quad (4.1.5)$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x} \left( s_x \frac{\partial V}{\partial \zeta} \right) = s_x \frac{\partial}{\partial \zeta} \left( \frac{\partial V}{\partial \zeta} \right) = s_x \frac{\partial V}{\partial \zeta} \frac{\partial}{\partial \zeta} \frac{\partial V}{\partial \zeta} = s_x^2 \frac{\partial^2 V}{\partial \zeta^2}. \quad (4.1.6)$$

In a similar way we find

$$\frac{\partial^2 V}{\partial y^2} = s_y^2 \frac{\partial^2 V}{\partial \zeta^2}; \quad \frac{\partial^2 V}{\partial z^2} = s_z^2 \frac{\partial^2 V}{\partial \zeta^2}. \quad (4.1.7)$$

When we substitute (4.1.6) and (4.1.7) in (4.0.1) and take into account that $s_x^2 + s_y^2 + s_z^2 = 1$, since $\hat{s}$ is a unit vector, the wave equation becomes

$$\frac{\partial^2 V}{\partial \zeta^2} - \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2} = 0. \quad (4.1.8)$$
Figure 4.1: A plane wave that propagates in direction $\hat{s}$, has no variation in any plane that is normal to $\hat{s}$.

By introducing two new variables $p$ and $q$, defined by

$$p = \zeta - vt ; \quad q = \zeta + vt,$$

we find (Exercise 2) that the wave equation in $(p, q)$ variables can be written

$$\frac{\partial^2 V}{\partial p \partial q} = 0.$$  (4.1.10)

This equation has the following general solution

$$V = V_1(p) + V_2(q),$$  (4.1.11)

where $V_1$ and $V_2$ are arbitrary functions. By substitution from (4.1.3) and (4.1.9) in (4.1.11), we find the following general plane-wave solution

$$V(r, t) = V_1(r \cdot \hat{s} - vt) + V_2(r \cdot \hat{s} + vt).$$  (4.1.12)

Note that

$$\zeta - vt = \zeta + v\tau - v(t + \tau),$$  (4.1.13)

so that

$$V_1(\zeta, t) = V_1(\zeta + v\tau, t + \tau).$$  (4.1.14)

Equation (4.1.14) shows that during the time $\tau$, $V_1$ is displaced a length $s = v\tau$ in the positive $\zeta$ direction, i.e. $V_1$ propagates with velocity $v$ in the positive $\zeta$ direction. The conclusion is that $V(\zeta \pm vt)$ represents a plane wave that propagates at velocity $v$ in the positive $\zeta$ direction (lower sign) or in the negative $\zeta$ direction (upper sign).

### 4.2 Spherical waves

Consider now solutions of the scalar wave equation (4.0.1) of the form

$$V = V(r, t),$$  (4.2.1)

where

$$r = |r| = \sqrt{x^2 + y^2 + z^2},$$  (4.2.2)
is the distance from the origin \((0,0,0)\). Since we have no angular dependence in this case, the Laplacian operator has the following form in spherical coordinates (Exercise 3)

\[
\nabla^2 V = \frac{1}{r} \frac{\partial^2}{\partial r^2} (rV),
\]

which upon substitution in the wave equation (4.0.1) gives

\[
\frac{\partial^2}{\partial r^2} (rV) - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} (rV) = 0.
\]

Since (4.2.4) is of the same form as (4.1.8), the solution becomes (cf. (4.1.12))

\[
rV = V_1 (r - vt) + V_2 (r + vt).
\]

Thus, we have obtained the following result: \(V(r \pm vt) \) represents a spherical wave that converges towards the origin (upper sign) or diverges away from the origin (lower sign). Thus, \(V(r+vt)\) propagates towards the origin with velocity \(v\), whereas \(V(r-vt)\) propagates away from the origin with velocity \(v\).

### 4.3 Harmonic (monochromatic) waves

At a given point \(\mathbf{r}\) in space the solution of the wave equation is a function only of time, i.e.

\[V(\mathbf{r}, t) = F(t),\]

where \(F(t)\) can be an arbitrary function. If \(F(t)\) has the simple form

\[F(t) = a \cos(\omega t - \delta),\]

then we have a harmonic wave in time. The quantities in (4.3.2) have the following meaning: \(a\) is the amplitude (positive), \(\omega\) is the angular frequency, and \(\omega t - \delta\) is the phase. A harmonic wave is also called a monochromatic wave because it consists of only one frequency or wavelength component.

The frequency \(\nu\) and the period \(T\) follow from

\[\nu = \frac{\omega}{2\pi} = \frac{1}{T}.
\]

The harmonic wave in (4.3.2) has period \(T\) because

\[F(t + T) = a \cos(\omega(t + T) - \delta) = a \cos(\omega t - \delta + 2\pi) = F(t).
\]

From (4.1.12) we see that the general expression for a wave that propagates in the \(\hat{s}\) direction can be written

\[V = V_1 (\mathbf{r} \cdot \hat{s} - vt) = V_1 \left[ -v \left( t - \frac{\mathbf{r} \cdot \hat{s}}{v} \right) \right] = V'_1 \left( t - \frac{\mathbf{r} \cdot \hat{s}}{v} \right),\]

where both \(V_1\) and \(V'_1\) are arbitrary functions. By replacing \(t\) with \(t - \frac{\mathbf{r} \cdot \hat{s}}{v}\) in (4.3.2) we get a harmonic plane wave

\[V(\mathbf{r}, t) = a \cos \left[ \omega \left( t - \frac{\mathbf{r} \cdot \hat{s}}{v} \right) + \delta \right] = a \cos[k r \cdot \hat{s} - \omega t + \delta],\]

where

\[k = \frac{\omega}{v},\]

is the wave number. We see that that \(V(\mathbf{r}, t)\) remains unchanged if we replace \(\mathbf{r} \cdot \hat{s}\) with \(\mathbf{r} \cdot \hat{s} + n\lambda\), where \(n = 1, 2, \ldots\), and \(\lambda\) is given by
The quantity $\lambda$ is called the *wavelength*. Note that for $t = \text{constant}$, $V(r, t)$ in (4.3.6) is periodic with wavelength $\lambda$, i.e.

$$V(r \cdot \hat{s}, t) = V(r \cdot \hat{s} + n\lambda, t) ; \quad n = 1, 2, 3, \ldots . \tag{4.3.9}$$

Now we introduce the *wave vector* or *propagation vector* $\mathbf{k}$, defined by

$$\mathbf{k} = k \hat{s}. \tag{4.3.10}$$

so that the expression (4.3.6) for a plane, harmonic wave can be written

$$V(r, t) = a \cos(k \cdot r - \omega t + \delta). \tag{4.3.11}$$

In a similar way the expression for a converging or a diverging harmonic spherical wave becomes

$$V(r, t) = a \frac{\cos(\mp kr - \omega t + \delta)}{r}, \tag{4.3.12}$$

where the upper sign corresponds to a converging spherical wave and the lower sign to a diverging spherical wave.

Consider now a plane, harmonic wave that propagates in the positive $z$ direction, so that [cf. (4.3.11)]

$$V(z, t) = a \cos(kz - \omega t + \delta). \tag{4.3.13}$$

A wave front is defined by the requirement that the phase shall be constant over it, i.e.

$$\phi = kz - \omega t + \delta = \text{constant}. \tag{4.3.14}$$

Hence it follows that on a wave front we have

$$z = vt + \text{constant} ; \quad v = \frac{\omega}{k}. \tag{4.3.15}$$

Thus, the wave front propagates at the velocity

$$v = \frac{\omega}{k}, \tag{4.3.16}$$

which is called the *phase velocity*.

### 4.4 Complex representation

Alternatively we can express (4.3.11) and (4.3.12) in the following way

$$V(r, t) = \text{Re}\{U(r)e^{-i\omega t}\}, \tag{4.4.1}$$

where Re{...} stands for the real part of {...}, and where the *complex amplitude* $U(r)$ is given by

$$U(r) = ae^{i(k \cdot r + \delta)}, \tag{4.4.2}$$

for a plane wave, and by

$$U(r) = a \frac{e^{i(\pm kr + \delta)}}{r}, \tag{4.4.3}$$

for a diverging (upper sign) or converging (lower sign) spherical wave.
Note that when we perform linear operations, such as differentiation, integration or summation, we can drop the 'Re' symbol during the operations, as long as we remember to take the real part of the result in the end.

By substituting
\[ V(r, t) = U(r) e^{-i\omega t}, \]
(4.4.4)
in the wave equation (4.0.1), we get
\[ (\nabla^2 + k^2)U(r) = 0, \]
(4.4.5)
which shows that the complex amplitude \( U(r) \) is a solution of the Helmholtz equation.

### 4.5 Linearity and the superposition principle

For any linear equation the sum of two or several solutions is also a solution. This is called the superposition principle. Since Maxwell's equations are linear, the superposition principle is valid for electromagnetic waves as long as the material equations are linear. The superposition principle implies that we can construct general solutions of the wave equation or Maxwell's equations by adding elementary solutions in the form of harmonic plane or spherical waves. We will discuss this in detail in Part II.

### 4.6 Phase velocity and group velocity

Consider a harmonic wave of the form [cf. (4.3.11)]
\[ V(r, t) = \text{Re} \left[ U(r) e^{-i\omega t} \right], \]
(4.6.1)
where the complex amplitude \( U(r) \) is a solution of the Helmholtz equation (4.4.5), i.e.
\[ (\nabla^2 + k^2)U(r) = 0. \]
(4.6.2)
The wave number \( k \) can be written
\[ k = \frac{\omega}{v} = \frac{\omega}{c} \left( \frac{c}{v} \right) = k_0 n, \]
(4.6.3)
where \( k_0 \) is the wave number in vacuum, i.e.
\[ k_0 = \frac{\omega}{c}, \]
(4.6.4)
and \( n \) is the refractive index given by
\[ n = \frac{c}{v} = \sqrt{\varepsilon \mu}. \]
(4.6.5)

A general wave \( V(r, t) \) can always be expressed as a sum of harmonic components. We will return to this later. If \( \varepsilon \) depends on \( \omega \), i.e. \( \varepsilon = \varepsilon(\omega) \), then the phase velocity also will depend on \( \omega \), since \( v = \frac{\omega}{k} = v(\omega) \). This means that different harmonic components will propagate at different phase velocities. A polychromatic wave or a pulse, which is comprised of many harmonic components, therefore will change its shape during propagation, and the energy will not propagate at the phase velocity, but at the group velocity, which is defined as
\[ v_g = \frac{d\omega}{dk}. \]
(4.6.6)
If \( n(\omega) = \text{constant} \), we have a non-dispersive medium. Since
\[ \omega = nk, \]
(4.6.7)
where the phase velocity $v = \frac{c}{n}$ now is constant, we have in this case

$$v_g = \frac{d}{dk}(vk) = v.$$  

Thus, the phase velocity and the group velocity are equal in a non-dispersive medium where $n = \text{constant}$. In dispersive media we have

$$v_g = \frac{d}{dk}(vk) = v + \frac{dv}{dk} = v + \frac{\lambda dv}{d\lambda} = v + v\frac{dv}{d\lambda},$$

where the last two results follow from the relation $k = \frac{2\pi}{\lambda} = \frac{2\pi}{v}.

### 4.7 Repetition

From Maxwell’s equations in source-free space ($J = 0; \rho = 0$) we find

$$\nabla^2 E - \frac{\varepsilon\mu}{c^2} \ddot{E} = 0; \quad \nabla^2 H - \frac{\varepsilon\mu}{c^2} \ddot{H} = 0. \quad (4.7.1)$$

Comparison of (4.7.1) with the scalar wave equation

$$\nabla^2 V - \frac{1}{v^2} \ddot{V} = 0, \quad (4.7.2)$$

shows that any Cartesian component of $E$ and $H$ satisfies the scalar wave equation with phase velocity $v$ given by

$$v = \frac{c}{\sqrt{\varepsilon\mu}} = \frac{c}{n}. \quad (4.7.3)$$

The scalar wave equation (4.7.2) has simple solutions in the form of plane waves or spherical waves.

#### Plane waves

For a plane wave $V$ is given by

$$V(r,t) = V_1(r \cdot \hat{s} - vt) + V_2(r \cdot \hat{s} - vt), \quad (4.7.4)$$

where $V(\zeta \mp vt)$ represents a plane wave that propagates in the positive $\zeta$ direction (upper sign) or in the negative $\zeta$ direction (lower sign).

#### Spherical waves

For a spherical wave $V$ is given by

$$V(r,t) = \frac{V_1(r - vt)}{r} + \frac{V_2(r + vt)}{r}, \quad (4.7.5)$$

where $\frac{V(r \mp vt)}{r}$ represents a spherical wave that propagates away from the origin (upper sign) or towards the origin (lower sign).

#### Harmonic (monochromatic) waves

A plane harmonic wave that propagates in the direction $k = k\hat{s}$ is given by

$$V(r,t) = a \cos(k \cdot r - \omega t + \delta), \quad (4.7.6)$$

and the corresponding spherical wave is

$$V(r,t) = \frac{a}{r} \cos(\pm kr - \omega t + \delta), \quad (4.7.7)$$

where the upper sign represents a diverging spherical wave and the lower sign represents a converging spherical wave.
Complex representation of harmonic waves

In complex notation we have

\[ V(r, t) = \text{Re}[U(r)e^{-i\omega t}]. \quad (4.7.8) \]

For a plane wave the complex amplitude \( U(r) \) is given by

\[ U(r) = ae^{i(k\cdot r + \delta)}, \]

and for a diverging or converging spherical wave it is given by

\[ U(r) = \frac{a}{r}e^{i(\pm kr + \delta)}. \]

By substituting (4.7.8) into the wave equation (4.7.2), we find that \( U(r) \) satisfies the Helmholtz equation, i.e.

\[ (\nabla^2 + k^2)U(r) = 0. \quad (4.7.9) \]
Chapter 5

Pulse propagation in a dispersive medium

Figure 5.1: A plane wave propagates in the positive $z$ direction in a dispersive medium that fills the half space $z \geq 0$.

Consider a polychromatic, plane wave that propagates in the positive $z$ direction in a linear, homogeneous, isotropic, and dispersive medium that fills the half space $z > 0$ (Fig. 5.1). The polychromatic, plane wave $u(z, t)$ is comprised of different harmonic components, which implies that we can represent $u(z, t)$ by the following inverse Fourier transform

$$u(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(z, \omega)e^{-i\omega t} d\omega,$$

where the frequency spectrum $\tilde{u}(z, \omega)$ is given as the Fourier transform of $u(z, t)$, i.e.

$$\tilde{u}(z, \omega) = \int_{-\infty}^{\infty} u(z, t)ei\omega t dt. \quad (5.2)$$

Thus, $u(z, t)$ and $\tilde{u}(z, \omega)$ constitute a Fourier transform pair. Since $\tilde{u}(z, \omega)$ can be any Cartesian component of the frequency spectrum of the electric or magnetic field, it satisfies the Helmholtz equation, i.e.

$$[\nabla^2 + k^2(\omega)]\tilde{u}(z, \omega) = 0, \quad (5.3)$$

where

$$k(\omega) = \frac{\omega}{v(\omega)} = \frac{\omega}{c} \frac{1}{v(\omega)} = \frac{\omega}{c} n(\omega). \quad (5.4)$$
Suppose now that $u(z,t)$ is known for all values of $t$ in the plane $z = 0$, and that $u(0,t)$ vanishes for $t < 0$.

Since there is no variation in the $x$ and $y$ directions, the Helmholtz equation (5.3) can be written as

$$\left[\frac{d^2}{dz^2} + k^2(\omega)\right]\tilde{u}(z,\omega) = 0,$$  

which has the following general solution

$$\tilde{u}(z,\omega) = u_+(\omega)e^{ik(\omega)z} + u_-(\omega)e^{-ik(\omega)z}. \quad (5.6)$$

If we consider propagation in the positive $z$ direction only, then $u_-(\omega) = 0$, so that (5.1) gives

$$u(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(0,\omega)e^{i\frac{z}{v}(\omega-t)}d\omega. \quad (5.7)$$

Now we put $z = 0$ in (5.7), take an inverse Fourier transform, and use (5.1) to obtain

$$u_+(\omega) = \int_{-\infty}^{\infty} u(0,\omega)e^{i\frac{z}{v}(\omega-t)}d\omega, \quad (5.8)$$

so that (5.7) gives

$$u(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(0,\omega)e^{i\frac{z}{v}(\omega-t)}d\omega, \quad (5.9)$$

or

$$u(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(0,\omega)e^{i\frac{z}{v}f(\omega)}d\omega, \quad (5.10)$$

where

$$f(\omega) = \omega[n(\omega) - \frac{v}{\omega}] \quad ; \quad \theta = \frac{ct}{z}. \quad (5.11)$$

Consider first the special case in which $n(\omega) = \frac{c}{\omega(\omega)} = \text{constant}$, which implies that we have a non-dispersive medium. Since $k = \frac{c}{v}$, where $v$ is now constant, we have from (5.1) and (5.9)

$$u(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(0,\omega)e^{-i\omega(-\frac{v}{v}+t)}d\omega = u\left(0, t - \frac{z}{v}\right). \quad (5.12)$$

This result shows that the pulse propagates in the positive $z$ direction at velocity $v$ without changing its shape.

Suppose now that the medium is dispersive and that the frequency spectrum $\tilde{g}(\omega)$ of the pulse in (5.10) does not contain singularities and that it is sufficiently wide. Then the main contribution to the pulse in (5.10) comes from frequencies $\omega_s$ for which the phase $f(\omega)$ in (5.11) is stationary, i.e. from $\omega_s$ that satisfy the equation

$$f'(\omega_s) = n'(\omega_s) - \theta + \omega_s n'(\omega_s) = 0. \quad (5.13)$$

A model that is commonly used to study propagation in dispersive media, is the so called Lorentz-medium. For such a medium with one single resonance frequency the refractive index $n(\omega)$ is given by the following expression

$$n(\omega) = \left[1 - \frac{b^2}{\omega^2 - \omega_0^2 + 2\delta i\omega}\right]^{1/2}, \quad (5.14)$$

where $b$ is a constant, $\omega_0$ is the resonance frequency, and $\delta$ represents the damping (attenuation) in the medium.
Equation (5.9) shows that when the medium is dispersive, then \( u(z, t) \) (for any \( z > 0 \)) is a sum of harmonic plane waves of the form \( \tilde{u}(0, \omega) \exp[i(k(\omega)z - \omega t)] \), where \( k_r(\omega) \) and \( k_i(\omega) \) are the real and the imaginary part, respectively, of \( k(\omega) \). Thus, the amplitude \( \tilde{u}(0, \omega) \exp[-k_i(\omega)z] \) is damped exponentially as \( z \) increases, and the phase velocity is given by \( v(\omega) = \frac{\omega}{k_r(\omega)} \), where \( k(\omega) = \left(\frac{\omega}{c}\right)n(\omega) = \left(\frac{\omega}{c}\right)[n_r(\omega) + in_i(\omega)] = k_r(\omega) + ik_i(\omega) \). Since the phase velocity \( v \) depends on the frequency \( \omega \), plane waves of different frequencies will arrive at a given position \( z \) at different times and thus cause a distortion of the pulse, i.e. the shape of the pulse will get changed. Also, the damping factor \( k_i(\omega) \) depends on \( \omega \), so that different frequency components will have different amplitudes when they arrive at a given position \( z \).
Chapter 6

General electromagnetic plane wave

A general electromagnetic plane wave can be written in the form

\[ E = E(\mathbf{k} \cdot \mathbf{r} - \omega t) \quad ; \quad H = H(\mathbf{k} \cdot \mathbf{r} - \omega t), \]

where \( \mathbf{k} = k\hat{s} \), with \( \hat{s} \) pointing in the direction of propagation. We introduce a new variable \( u = \mathbf{k} \cdot \mathbf{r} - \omega t \), so that

\[ \frac{\partial u}{\partial x} = k_x \quad ; \quad \frac{\partial u}{\partial y} = k_y \quad ; \quad \frac{\partial u}{\partial z} = k_z \quad ; \quad \frac{\partial u}{\partial t} = -\omega. \]  

In source-free space Maxwell’s equations (1.1.1)-(1.1.2) are given by

\[ \nabla \times \mathbf{H} = \frac{1}{c} \dot{\mathbf{D}} = \frac{\varepsilon}{c} \dot{\mathbf{E}}, \]

\[ \nabla \times \mathbf{E} = -\frac{1}{c} \dot{\mathbf{B}} = \frac{\mu}{c} \dot{\mathbf{H}}. \]

By using the chain rule, we find that the \( x \) component of \( \nabla \times \mathbf{E} \) can be expressed as follows

\[ (\nabla \times \mathbf{E})_x = \nabla_y E_z - \nabla_z E_y = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = \frac{dE_z}{du} \frac{\partial u}{\partial y} - \frac{dE_y}{du} \frac{\partial u}{\partial z} \]

\[ = E'_z k_y - E'_y k_z = (\mathbf{k} \times \mathbf{E}')_x = \frac{\omega}{v} (\hat{s} \times \mathbf{E}')_x, \]

where \( \mathbf{E}' = \frac{dE}{du} \). By proceeding in a similar manner, we find that

\[ \nabla \times \mathbf{E} = \frac{\omega}{v} \hat{s} \times \mathbf{E}', \]

\[ \nabla \times \mathbf{H} = \frac{\omega}{v} \hat{s} \times \mathbf{H}', \]

where

\[ \mathbf{E}' = \frac{d\mathbf{E}}{du} ; \quad \mathbf{H}' = \frac{d\mathbf{H}}{du}. \]

Further, we have

\[ \dot{\mathbf{E}} = \frac{\partial \mathbf{E}}{\partial t} = \frac{d\mathbf{E}}{du} \frac{\partial u}{\partial t} = -\omega \mathbf{E}' ; \quad \dot{\mathbf{H}} = -\omega \mathbf{H}'. \]
By substitution of (6.6)-(6.9) into Maxwell’s equations (6.3)-(6.4) the result is

\[ \hat{s} \times H' = \frac{\varepsilon}{c} (-v) E' = -\frac{\varepsilon}{c} \frac{c}{\sqrt{\varepsilon \mu}} E' = -\sqrt{\varepsilon \mu} E', \]  

(6.10)

\[ \hat{s} \times E' = -\frac{\mu}{c} (-v) H' = \sqrt{\frac{\varepsilon}{\mu}} H', \]  

(6.11)

where we have used (3.9). Thus, we have

\[ E' = -\sqrt{\frac{\mu}{\varepsilon}} \hat{s} \times H ;; H' = \sqrt{\frac{\varepsilon}{\mu}} \hat{s} \times E'. \]  

(6.12)

By integrating over \( u \) in (6.12) and setting the integration constant equal to zero, we get

\[ E = -\sqrt{\frac{\mu}{\varepsilon}} \hat{s} \times H ;; H = \sqrt{\frac{\varepsilon}{\mu}} \hat{s} \times E. \]  

(6.13)

Scalar multiplication of the equations in (6.13) with \( \hat{s} \) gives

\[ \hat{s} \cdot E = \hat{s} \cdot H = 0, \]  

(6.14)

which shows that both \( E \) and \( H \) are transverse waves, i.e. both \( E \) and \( H \) are normal to the propagation direction \( \hat{s} \), as illustrated in Fig. 6.1. Thus, the vectors \( \hat{s}, E, \) and \( H \) represent a right-handed Cartesian co-ordinate system.

![Figure 6.1](image)

Figure 6.1: The vectors \( E, H, \) and \( \hat{s} \) for an electromagnetic plane wave represent a right-handed Cartesian co-ordinate system.

For the electric and the magnetic energy density we find

\[ w_e = \frac{1}{8 \pi} E \cdot D = \frac{\varepsilon}{8 \pi} E^2 ;; E = |E|, \]  

(6.15)

\[ w_m = \frac{1}{8 \pi} B \cdot H = \frac{\mu}{8 \pi} H^2 ;; H = |H|. \]  

(6.16)

Since \( \sqrt{\mu} H = \sqrt{\varepsilon} E \) (cf. (6.13)), we get \( w_e = w_m \), and the total energy density becomes

\[ w = w_e + w_m = 2w_e = \frac{1}{4 \pi} \varepsilon E^2 = 2w_m = \frac{1}{4 \pi} \mu H^2, \]  

(6.17)

and the Poynting vector (2.4) becomes

\[ S = \frac{c}{4 \pi} E \times H = \frac{c}{4 \pi} EH \hat{s} = \frac{c}{4 \pi} E \sqrt{\frac{\mu}{\varepsilon}} E \hat{s} = \left( \frac{1}{4 \pi} \varepsilon E^2 \right) \left( \frac{c}{\sqrt{\varepsilon \mu}} \right) \hat{s} = w \hat{s}. \]  

(6.18)
Thus, we have

$$S = wv \hat{s}, \quad (6.19)$$

which shows that the Poynting vector represents the energy flow, both with respect to absolute value and direction. A dimensional analysis of (6.19) shows that

$$[|S|] = [w][v] = \text{Energie} \cdot \frac{m}{s} = \text{Energie} \cdot \frac{m^2}{s} = \frac{W}{m^2}. \quad (6.20)$$

Thus, $S$ represents the amount of energy per unit time that passes through a unit area of the plane that is spanned by $E$ and $H$, as asserted previously in chapter 2.
Chapter 7

Harmonic electromagnetic waves of arbitrary form - Time averages

The \( \mathbf{E} \) and \( \mathbf{H} \) fields for a harmonic wave of arbitrary form can be written

\[
\mathbf{E}(\mathbf{r}) = \text{Re} \left\{ \mathbf{E}_0(\mathbf{r}) e^{-i\omega t} \right\} ; \quad \mathbf{H}(\mathbf{r}) = \text{Re} \left\{ \mathbf{H}_0(\mathbf{r}) e^{-i\omega t} \right\},
\]

where \( \mathbf{E}_0(\mathbf{r}) \) and \( \mathbf{H}_0(\mathbf{r}) \) are complex vectors. Thus, we have

\[
\mathbf{E}_0(\mathbf{r}) = \mathbf{E}_R^0(\mathbf{r}) + i \mathbf{E}_I^0(\mathbf{r}),
\]

\[
\mathbf{H}(\mathbf{r}) = \mathbf{H}_R^0(\mathbf{r}) + i \mathbf{H}_I^0(\mathbf{r}),
\]

where \( \mathbf{E}_R^0, \mathbf{E}_I^0, \mathbf{H}_R^0, \) and \( \mathbf{H}_I^0 \) are real vectors.

Since optical frequencies are very high \( (\omega \approx 10^{15} \text{ s}^{-1}) \), we can only observe averages of \( \mathcal{W}_c, \mathcal{W}_m, \) and \( \mathcal{S} \), taken over a time interval \( -T' \leq t \leq T' \), where \( T' \) is much larger than the period \( T = \frac{2\pi}{\omega} \).

For the time average of the electric energy density we have [cf. (2.1)]

\[
\langle \mathcal{W}_c \rangle = \frac{1}{T'} \int_{-T'}^{T'} \frac{\varepsilon}{8\pi} \mathbf{E}^2 dt.
\]

For any complex number \( z \), we have \( \text{Re} z = \frac{1}{2}(z + z^*) \), where \( z^* \) is the complex conjugate of \( z \). Therefore, we may write

\[
\mathbf{E} = \text{Re} [\mathbf{E}_0(\mathbf{r}) e^{-i\omega t}] = \frac{1}{2} [\mathbf{E}_0 e^{-i\omega t} + \mathbf{E}_0^* e^{+i\omega t}],
\]

so that we get

\[
|\mathbf{E}|^2 = \mathbf{E} \cdot \mathbf{E} = \frac{1}{4} [\mathbf{E}_0 e^{-i\omega t} + \mathbf{E}_0^* e^{+i\omega t}] \cdot [\mathbf{E}_0 e^{-i\omega t} + \mathbf{E}_0^* e^{+i\omega t}] = \frac{1}{4} [\mathbf{E}_0^2 e^{-2i\omega t} + 2 \mathbf{E}_0 \cdot \mathbf{E}_0^* + \mathbf{E}_0^* \cdot \mathbf{E}_0 e^{2i\omega t}].
\]

Further, we have

\[
\frac{1}{2T'} \int_{-T'}^{T'} e^{-2i\omega t} dt = \frac{1}{2T'} \left[ \left. \frac{e^{-2i\omega t}}{-2i\omega} \right|_{-T'}^{T'} = \frac{1}{2T'} \left( \frac{1}{\omega} \right) \sin(2\omega T') = \frac{1}{4\pi} \frac{T}{T'} \sin(2\omega T'). \right.
\]

Since \( T' \gg T \), the integral that includes the factor \( e^{-2i\omega t} \) can be neglected. Similarly, the integral that includes the factor \( e^{2i\omega t} \) can be neglected, and we get

\[
\langle \mathcal{W}_c \rangle = \frac{\varepsilon}{16\pi} \mathbf{E}_0 \cdot \mathbf{E}_0^*.
\]

By proceeding in a similar manner, we find that the time average of the magnetic energy density becomes

\[ \langle w_m \rangle = \frac{\mu}{16\pi} \mathbf{H}_0 \cdot \mathbf{H}_0^* \]  

(7.8)

The time average of the Poynting vector is given by [cf. (2.4)]

\[ \langle \mathbf{S} \rangle = \frac{1}{2T'} \int_{-T'}^{T'} \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H}) dt, \]  

(7.9)

where \( \mathbf{E} \times \mathbf{H} \) can be written

\[
\mathbf{E} \times \mathbf{H} = \frac{1}{2} [\mathbf{E}_0 e^{-i\omega t} + \mathbf{E}_0^* e^{i\omega t}] \times \frac{1}{2} [\mathbf{H}_0 e^{-i\omega t} + \mathbf{H}_0^* e^{i\omega t}]
\]

\[ = \frac{1}{4} (\mathbf{E}_0 \times \mathbf{H}_0 e^{-2i\omega t} + \mathbf{E}_0 \times \mathbf{H}_0^* + \mathbf{E}_0^* \times \mathbf{H}_0 + \mathbf{E}_0^* \times \mathbf{H}_0^* e^{2i\omega t}). \]  

(7.10)

By substituting (7.10) into (7.9) and performing time averaging, we find that the time average of the Poynting vector becomes

\[ \langle \mathbf{S} \rangle = \frac{c}{16\pi} (\mathbf{E}_0 \times \mathbf{H}_0^* + \mathbf{E}_0^* \times \mathbf{H}_0) = \frac{c}{8\pi} \text{Re}(\mathbf{E}_0 \times \mathbf{H}_0^*). \]  

(7.11)
Chapter 8

Harmonic electromagnetic plane wave – Polarisation

For an electromagnetic plane wave that is time harmonic, each Cartesian component of \( \mathbf{E} \) and \( \mathbf{H} \) is of the form

\[
a \cos(\tau + \delta) = \text{Re}[ae^{i(\tau + \delta)}] \quad a > 0,
\]

where

\[
\tau = k \cdot r - \omega t.
\]

Let the \( z \) axis point in the \( \hat{s} \) direction. Then only the \( x \) and \( y \) components of \( \mathbf{E} \) and \( \mathbf{H} \) are non-zero, since the electromagnetic field of a plane wave is transverse.

Now we want to determine that curve which the end point of the electric vector describes during propagation. This curve consists of points that have co-ordinates \((E_x, E_y)\) given by

\[
E_x = a_1 \cos(\tau + \delta_1) \quad a_1 > 0,
\]

\[
E_y = a_2 \cos(\tau + \delta_2) \quad a_2 > 0,
\]

\[
E_z = 0.
\]

In order to determine that curve which \( \mathbf{E}(\tau) \) describes (Fig. 8.1), we eliminate \( \tau \) from (8.3)-(8.4). We let \( \beta = \tau + \delta_1 \) and get

\[
E_x = a_1 \cos \beta,
\]

\[
E_y = a_2 \cos(\beta + \delta) = a_2 [\cos \beta \cos \delta - \sin \beta \sin \delta],
\]

where \( \delta = \delta_2 - \delta_1 \). We substitute from (8.6) into (8.7) and get

\[
\frac{E_y}{a_2} = \frac{E_x}{a_1} \cos \delta - \sqrt{1 - \left(\frac{E_x}{a_1}\right)^2} \sin \delta,
\]

which upon squaring gives

\[
\left(\frac{E_x}{a_1}\right)^2 + \left(\frac{E_y}{a_2}\right)^2 - 2 \frac{E_x}{a_1} \frac{E_y}{a_2} \cos \delta = \sin^2 \delta.
\]

This is the equation of a conic section. The cross term implies that it is rotated relative to the co-ordinate axes \((x, y)\). By letting \( \delta = \frac{\pi}{2} \), we get
Figure 8.1: Instantaneous picture of the electric vector of a plane wave that propagates in the $z$ direction.

\[
\left( \frac{E_x}{a_1} \right)^2 + \left( \frac{E_y}{a_2} \right)^2 = 1,
\]
where $a_1$ and $a_2$ are constants. This equation describes an ellipse. In a coordinate system $(\xi, \eta)$, which coincides with the axes of the ellipse, the equations for the field components become

\[
E_\xi = a \cos(\tau + \delta_0),
\]
\[
E_\eta = \pm b \sin(\tau + \delta_0),
\]
which upon squaring gives

\[
\left( \frac{E_\xi}{a} \right)^2 + \left( \frac{E_\eta}{b} \right)^2 = 1.
\]

When $\tau + \delta_0 = 0$, we have $E_\xi = a$; $E_\eta = 0$, and when $\tau + \delta_0 = \frac{\pi}{2}$, we have $E_\xi = 0$; $E_\eta = \pm b$. This shows that when the upper or lower sign in (8.12) applies, the electric vector rotates against or with the clock, respectively, if we view the $xy$ plane from the positive $z$ axis. Rotation against the clock is called left-handed polarisation, and rotation with the clock is called right-handed polarisation.

The relation between the two coordinate systems $(x, y)$ and $(\xi, \eta)$ is shown in Fig. 8.2, where (cf. Exercise 7)

\[
a^2 + b^2 = a_1^2 + b_2^2,
\]
\[
\tan 2\psi = \tan(2\alpha) \cos \delta \quad ; \quad \tan \alpha = \frac{a_2}{a_1} \quad (0 \leq \alpha \leq \frac{\pi}{2}),
\]
\[
\sin 2\psi = \sin(2\alpha) \sin \delta \quad ; \quad \tan \psi = \pm \frac{b}{a}.
\]

Since $\sin \delta < 0$ when the upper sign in (8.12) applies, we have left-handed polarisation when $\sin \delta < 0$.

We consider now some special cases of (8.6)-(8.7).

**Linear polarisation.** If the phase difference $\delta$ is equal to an integer times $\pi$, i.e. if

\[
\delta = m\pi \quad (m = 1, \pm 1, \pm 2, \ldots),
\]
then we get from (8.6)-(8.7)

\[
E_x = a_1 \cos \beta,
\]
Figure 8.2: The end point of the electric vector describes an ellipse that is inscribed in a rectangle with sides $2a_1$ and $2a_2$.

$$E_y = a_2 \cos(\beta + m\pi) = a_2(-1)^m \frac{E_x}{a_1},$$  \hspace{1cm} (8.19)

which shows that the ellipse degenerates into a straight line, i.e.

$$\frac{E_y}{E_x} = (-1)^m \frac{a_2}{a_1}.$$ \hspace{1cm} (8.20)

**Circular polarisation.** If the amplitudes are equal and the phase difference is $\pm \frac{\pi}{2}$ plus a multiple of $2\pi$, i.e. if

$$a_1 = a_2,$$  \hspace{1cm} (8.21)

$$\delta = \pm \frac{\pi}{2} + 2m\pi \hspace{0.5cm} (m = 0, \pm 1, \pm 2, \ldots),$$  \hspace{1cm} (8.22)

then the ellipse in (8.6)-(8.7) degenerates into a circle, i.e.

$$E_x = a \cos \beta,$$  \hspace{1cm} (8.23)

$$E_y = a \cos \left(\beta + 2m\pi \pm \frac{\pi}{2}\right) = \mp a \sin \beta.$$  \hspace{1cm} (8.24)

By squaring these two equations, we get

$$E_x^2 + E_y^2 = a^2.$$  \hspace{1cm} (8.25)

We have right-handed circular polarisation when $E_y = -a \sin \beta$ and left-handed circular polarisation when $E_y = +a \sin \beta$. 
Chapter 9

Reflection and refraction of a plane wave

Figure 9.1: Reflection and refraction of a plane wave at a plane interface between two different media. Illustration of propagation directions and angles of incidence, reflection, and transmission.

We let a plane wave be incident upon a plane interface between two different media, as shown in Fig. 9.1. The incident wave gives rise to a reflected wave and a transmitted wave, which we assume are plane waves as well. Thus, each component of $E$ or $H$ can be written

$$A_q^j = \text{Re} \{a_q^j e^{i(k^q \cdot r - \omega t)}\} \quad (j = x, y, z),$$

(9.0.1)

where $A$ stands for $E$ or $H$ and $q = i, r, t$, so that $k^i, k^r, \text{and } k^t$ are the wave vectors of the incident, reflected, and transmitted waves, respectively.

9.1 Reflection law and refraction law (Snell’s law)

The existence of continuity conditions that $E$ and $H$ must satisfy at the interface between the two media in Fig. 9.1, implies that when $r$ represents a point at the interface, the argument in the exponential function in (9.0.1) must be the same for the reflected and transmitted waves as for the incident wave. Thus, we have
\[ \mathbf{k}^i \cdot \mathbf{r} - \omega t = \mathbf{k}^r \cdot \mathbf{r} - \omega t = \mathbf{k}^f \cdot \mathbf{r} - \omega t, \]  

(9.1.1)

or

\[ \mathbf{k}^i \cdot \mathbf{r} = \mathbf{k}^r \cdot \mathbf{r} = \mathbf{k}^f \cdot \mathbf{r}. \]  

(9.1.2)

Now we introduce a Cartesian co-ordinate system in which the unit vectors \( \mathbf{n} \), \( \mathbf{b} \), and \( \mathbf{t} \) represent a right-handed system (Fig. 9.2). Let \( \mathbf{n} \) point along the interface normal into the medium of the refracted wave, and let \( \mathbf{b} \) and \( \mathbf{t} \) be defined by

\[ \mathbf{b} = \frac{\mathbf{k}^i \times \mathbf{n}}{|\mathbf{k}^i \times \mathbf{n}|} ; \quad \mathbf{t} = \mathbf{n} \times \mathbf{b}. \]  

(9.1.3)

In this co-ordinate system we have

\[ \mathbf{k}^i = k^i_t \mathbf{t} + k^i_n \mathbf{n} ; \quad k^i_n = \mathbf{k}^i \cdot \mathbf{b} = 0, \]  

(9.1.4)

\[ \mathbf{k}^r = k^r_t \mathbf{t} + k^r_n \mathbf{n} + k^r_b \mathbf{b}, \]  

(9.1.5)

\[ \mathbf{k}^f = k^f_t \mathbf{t} + k^f_n \mathbf{n} + k^f_b \mathbf{b}, \]  

(9.1.6)

\[ \mathbf{r} = r_t \mathbf{t} + r_b \mathbf{b}. \]  

(9.1.7)

Note that the co-ordinate system is defined such that \( \mathbf{k}^f \) has no component along \( \mathbf{b} \), i.e. \( \mathbf{b} \) is normal to the plane of incidence, which is spanned by the vectors \( \mathbf{k}^i \) and \( \mathbf{n} \).

Since

\[ \mathbf{k}^i \cdot \mathbf{r} = (k^i_t \mathbf{t} + k^i_n \mathbf{n}) \cdot (r_t \mathbf{t} + r_b \mathbf{b}) = k^i_t r_t, \]  

(9.1.8)

\[ \mathbf{k}^r \cdot \mathbf{r} = (k^r_t \mathbf{t} + k^r_n \mathbf{n} + k^r_b \mathbf{b}) \cdot (r_t \mathbf{t} + r_b \mathbf{b}) = k^r_t r_t + k^r_b r_b, \]  

(9.1.9)

\[ \mathbf{k}^f \cdot \mathbf{r} = (k^f_t \mathbf{t} + k^f_n \mathbf{n} + k^f_b \mathbf{b}) \cdot (r_t \mathbf{t} + r_b \mathbf{b}) = k^f_t r_t + k^f_b r_b, \]  

(9.1.10)

it follows from the continuity condition (9.1.2) that

\[ k^i_t r_t = k^r_t r_t + k^f_t r_t = k^f_t r_t + k^r_b r_b. \]  

(9.1.11)
But since (9.1.11) shall apply to any point at the interface, i.e. to all values of \( r_t \) and \( r_b \), we must have

\[ k_r^t = k_b^t = 0. \]  

(9.1.12)

Thus, both \( k^r \) and \( k^t \) must lie in the plane of incidence spanned by \( k^i \) and \( \hat{n} \). Therefore, we have

\[ k_t^i = k_t^i = k_t^t = k_t, \]  

(9.1.13)

which implies that the components of \( k^i \), \( k^r \), and \( k^t \) parallel to the interface are equal. By using

\[ \hat{n} \times k^q = \hat{n} \times (k_t \hat{t} + k_n \hat{n}) = -\hat{b}k_t; \quad q = i, r, t, \]  

(9.1.14)

we find that

\[ \hat{n} \times k^i = \hat{n} \times k^r, \]  

(9.1.15)

\[ \hat{n} \times k^t = \hat{n} \times k^r. \]  

(9.1.16)

Further, we use the relation \(|a \times b| = |a||b|\sin \theta\), where \( \theta \) is the angle between the vectors \( a \) and \( b \). Thus, we find from (9.1.15) and Fig. 9.1 that

\[ k^i \sin \theta^i = k^r \sin \theta^r. \]  

(9.1.17)

Also, we know that \( k^i = k^r = n_1 k_0 \), where \( n_1 \) is the refractive index in medium 1, and \( k_0 \) is the wave number in vacuum. The reflection law therefore becomes

\[ \theta^i = \theta^r, \]  

(9.1.18)

which in (9.1.15) is given in vectorial form.

From (9.1.16) and Fig. 9.1 we get the refraction law or Snell’s law

\[ k^i \sin \theta^i = k^t \sin \theta^t. \]  

(9.1.19)

which by using \( k^i = n_1 k_0 \) and \( k^t = n_2 k_0 \), becomes

\[ n_1 \sin \theta^i = n_2 \sin \theta^t. \]  

(9.1.20)

Equation (9.1.16) represents Snell’s law in vector form. Note that (9.1.15) and (9.1.16) contain more information than (9.1.18) and (9.1.20). From the vector equations it is clear that \( k^r \) and \( k^t \) lie in the plane of incidence.

### 9.2 Generalisation of the reflection law and Snell’s law

The reflection law and Snell’s law (the refraction law) can be generalised to include non-planar waves that are incident upon a non-planar interface. This is illustrated in Fig. 9.3, where the field from a point source propagates towards a curved interface. Suppose now that the distance from the point source to the interface is much larger than the wavelength. Then at each point on the interface we may consider the incident wave to be a plane wave \( \text{locally} \), and we may replace the interface \( \text{locally} \) by the tangent plane through the point in question. Then we can use Snell’s law and the reflection law as derived for a plane wave that is incident upon a plane interface, as illustrated in Fig. 9.3.
9.3 Reflection and refraction of plane electromagnetic waves

Note that the reflection law and the refraction law apply to all types of plane waves, i.e. to acoustic, electromagnetic, and elastic waves. In the derivation we have only used that $k^q \cdot r - \omega t$ ($q = i, r, t$) shall be the same for $q = i$, $q = r$, and $q = t$. Now we take a closer look at the reflection and refraction of plane electromagnetic waves in order to determine how much of the energy in the incident wave that is reflected and transmitted.

We know that a plane electromagnetic wave is transverse, i.e. that both $E$ and $B = \mu H$ are normal to the propagation direction $k = k \hat{s}$. In Fig. 9.1 we have chosen the $z$ axis in the direction of the interface normal. If $E$ is normal to the plane of incidence, we have $s$ polarisation (from German, “Senkrecht”) or $TE$ polarisation (“transverse electric” relative to the plane of incidence or the $z$ axis). And if $E$ is parallel with the plane of incidence, we have $p$ polarisation or $TM$ polarisation, since in this case $B$ is normal to the plane of incidence or the $z$ axis; hence the use of the term $TM$ or “transverse magnetic”.

A general time-harmonic, plane electromagnetic wave consists of both a $TE$ and a $TM$ component. With the time dependence $e^{-i\omega t}$ suppressed, we have for the spatial part of the field

$$E = E^{TE} + E^{TM}; \quad B = B^{TE} + B^{TM},$$

(9.3.1)

$$E^{TE} = E^{TE} \frac{k_t \times \hat{e}_z}{k_t} e^{ik_t r},$$

(9.3.2)

$$E^{TM} = E^{TM} \frac{k \times (k_t \times \hat{e}_z) \times \hat{e}_z}{k k_t} e^{ik_t r},$$

(9.3.3)

$$B^{TE} = \frac{1}{k_0} k \times E^{TE} = E^{TE} \frac{k \times (k_t \times \hat{e}_z)}{k_0 k_t} e^{ik_t r},$$

(9.3.4)

$$B^{TM} = \frac{1}{k_0} k \times E^{TM} = E^{TM} \frac{1}{k_0 k k_t} k \times [k \times (k_t \times \hat{e}_z)] e^{ik_t r}.$$  

(9.3.5)

But since $k \times [k \times (k_t \times \hat{e}_z)] = k [k \cdot (k_t \times \hat{e}_z)] - (k_t \times \hat{e}_z) k \cdot k = -k^2 k_t \times \hat{e}_z$, we get

$$B^{TM} = \frac{-k}{k_0} E^{TM} \frac{k_t \times \hat{e}_z}{k_t} e^{ik_t r}.$$ 

(9.3.6)
Note that the vectors
\[ \hat{e}^{TE} = \frac{k \times \hat{e}_z}{k_t}; \quad \hat{e}^{TM} = \frac{k \times (k_t \times \hat{e}_z)}{kk_t}, \]
are unit vectors in the directions of \( \mathbf{E}^{TE} \) and \( \mathbf{E}^{TM} \), respectively.

We represent each of the incident, reflected, and transmitted fields in the manner given above, so that \((q = i, r, t)\)
\[ \mathbf{E}^q = \mathbf{E}^{TEq} + \mathbf{E}^{TMq}; \quad \mathbf{B}^q = \mathbf{B}^{TEq} + \mathbf{B}^{TMq}, \]
(9.3.8)
\[ \mathbf{E}^{TEq} = \mathbf{E}^{TE0} \frac{k \times \hat{e}_z}{k_t} e^{ik \cdot r}, \]
(9.3.9)
\[ \mathbf{E}^{TMq} = \mathbf{E}^{TM0} \frac{k \times (k_t \times \hat{e}_z)}{kk_t} e^{ik \cdot r}, \]
(9.3.10)
\[ \mathbf{B}^{TEq} = \mathbf{B}^{TE0} \frac{k^q}{k_0} \mathbf{E}^{TE0} k \times (k_t \times \hat{e}_z) e^{ik \cdot r}, \]
(9.3.11)
\[ \mathbf{B}^{TMq} = -\frac{k^q}{k_0} \mathbf{E}^{TM0} k \times \hat{e}_z e^{ik \cdot r}, \]
(9.3.12)
where
\[ k^i = k_t + k_z \hat{e}_z; \quad k_t = k_x \hat{e}_x + k_y \hat{e}_y, \]
(9.3.13)
\[ k^r = k_t - k_z \hat{e}_z; \quad k^t = k_t + k_z \hat{e}_z, \]
(9.3.14)
\[ k^q = \begin{cases} k_1 = n_1 k_0 & \text{for } q = i, r \\ k_2 = n_2 k_0 & \text{for } q = t. \end{cases} \]
(9.3.15)

The continuity conditions that must be satisfied at the interface \( z = 0 \) are that the tangential components of \( \mathbf{E} \) and \( \mathbf{H} = \frac{1}{\mu} \mathbf{B} \) be continuous, i.e.
\[ \hat{e}_z \times \left\{ \mathbf{E}^{TEi} + \mathbf{E}^{TEr} - \mathbf{E}^{TEt} + \mathbf{E}^{TMi} + \mathbf{E}^{TMr} - \mathbf{E}^{TMt} \right\} = 0, \]
(9.3.16)
\[ \hat{e}_z \times \left\{ \frac{1}{\mu_1} (\mathbf{B}^{TEi} + \mathbf{B}^{TEr}) - \frac{1}{\mu_2} \mathbf{B}^{TEt} + \frac{1}{\mu_1} (\mathbf{B}^{TMi} + \mathbf{B}^{TMr}) - \frac{1}{\mu_2} \mathbf{B}^{TMt} \right\} = 0. \]
(9.3.17)
Further, we have
\[ \hat{e}_z \times [k^q \times (k_t \times \hat{e}_z)] = (k^q \cdot \hat{e}_z) \hat{e}_z \times k_t, \]
(9.3.18)
\[ \hat{e}_z \times (k_t \times \hat{e}_z) = k_z. \]
(9.3.19)
By substituting from (9.3.9)-(9.3.12) into the boundary conditions (9.3.16)-(9.3.17) and using (9.1.2) and (9.3.18)-(9.3.19), we get
\[ k_t \left\{ E^{TEi} + E^{TEr} - E^{TEt} \right\} + \hat{e}_z \times k_t \left\{ \frac{k_z}{k_1} E^{TMi} - \frac{k_z}{k_1} E^{TEr} - \frac{k_z}{k_2} E^{TMt} \right\} = 0, \]
(9.3.20)
\[ \hat{e}_z \times k_t \left\{ \frac{1}{\mu_1} \left( \frac{k_z}{k_0} E^{TEi} - \frac{k_z}{k_0} E^{TEr} \right) - \frac{1}{\mu_2} \frac{k_z}{k_0} E^{TEt} \right\} + k_t \left\{ \frac{1}{\mu_1} \left( \frac{-k_z}{k_0} E^{TMi} - \frac{k_z}{k_0} E^{TMr} \right) - \frac{k_z}{k_0} E^{TMt} \right\} = 0. \]
(9.3.21)
Since \( \mathbf{k}_r \) and \( \hat{\mathbf{e}}_z \times \mathbf{k}_r \) are orthogonal vectors, the expression inside each of the \{ \} parentheses in (9.3.20) and (9.3.21) must vanish, i.e.

\[
E^{TEi} + E^{TEr} = E^{TEi}, \tag{9.3.22}
\]

\[
k_{z1}\mu_2 \left( E^{TEi} - E^{TEr} \right) = k_{z2}\mu_1 E^{TEi}, \tag{9.3.23}
\]

\[
k_{z1}k_2 \left( E^{TMi} - E^{TMr} \right) = k_{z2}k_1 E^{TMi}, \tag{9.3.24}
\]

\[
k_1\mu_2 \left( E^{TMi} - E^{TMr} \right) = k_2\mu_1 E^{TMi}. \tag{9.3.25}
\]

Now we define reflection and transmission coefficients as

\[
R^{TE} = \frac{E^{TEr}}{E^{TEi}}; \quad T^{TE} = \frac{E^{TEi}}{E^{TEi}}, \tag{9.3.26}
\]

\[
R^{TM} = \frac{E^{TMr}}{E^{TMi}}; \quad T^{TM} = \frac{E^{TMi}}{E^{TMi}}, \tag{9.3.27}
\]

so that (9.3.22)-(9.3.25) give

\[
1 + R^{TE} = T^{TE}; \quad 1 - R^{TE} = \frac{k_{z2}\mu_1 T^{TE}}{k_{z1}\mu_2}, \tag{9.3.28}
\]

\[
1 - R^{TM} = \frac{k_{z2}k_1 T^{TM}}{k_{z1}k_2}; \quad 1 + R^{TM} = \frac{k_2\mu_1 T^{TM}}{k_1\mu_2}. \tag{9.3.29}
\]

The two equations in (9.3.28) have the following solution

\[
R^{TE} = \frac{\mu_2k_{z1} - \mu_1k_{z2}}{\mu_2k_{z1} + \mu_1k_{z2}}, \quad T^{TE} = \frac{2\mu_2k_{z1}}{\mu_2k_{z1} + \mu_1k_{z2}}, \tag{9.3.30}
\]

whereas the two equations in (9.3.29) give

\[
R^{TM} = \frac{k_2^2\mu_1k_{z1} - k_1^2\mu_2k_{z2}}{k_2^2\mu_1k_{z1} + k_1^2\mu_2k_{z2}}, \quad T^{TM} = \frac{2k_1\mu_2k_{z1}}{k_2^2\mu_1k_{z1} + k_1^2\mu_2k_{z2}}. \tag{9.3.31}
\]

The interpretation of the reflection and transmission coefficients follow from (9.3.26)-(9.3.27). Thus, the reflection coefficient represents the amplitude ratio between the reflected and the incident \( \mathbf{E} \) field, whereas the transmission coefficient represents the amplitude ratio between the transmitted and the incident \( \mathbf{E} \) field.

Note that (9.3.22)-(9.3.23) and (9.3.28) contain only \( TE \) quantities, whereas equations (9.3.24)-(9.3.25) and (9.3.29) contain only \( TM \) quantities. This implies that these two wave types are independent or de-coupled upon reflection and refraction. Thus, an incident \( TE \) plane wave produces a reflected \( TE \) plane wave and a transmitted \( TE \) plane wave, whereas an incident \( TM \) plane wave produces a reflected \( TM \) plane wave and a transmitted \( TM \) plane wave. Upon reflection and refraction there is no coupling between \( TE \) and \( TM \) waves.

From Fig. 9.1 it follows that

\[
k_{z1} = k^1 \cdot \hat{\mathbf{e}}_z = k_1 \cos \theta^i ; \quad k_{z2} = k^1 \cdot \hat{\mathbf{e}}_z = k_2 \cos \theta^i, \tag{9.3.32}
\]

so that if \( \mu_1 = \mu_2 = 1 \) the reflection and transmission coefficients become

\[
T^{TM} = \frac{2n_1 \cos \theta^i}{n_2 \cos \theta^i + n_1 \cos \theta^i}; \quad R^{TM} = \frac{n_2 \cos \theta^i - n_1 \cos \theta^i}{n_2 \cos \theta^i + n_1 \cos \theta^i}, \tag{9.3.33}
\]

\[
T^{TE} = \frac{2n_1 \cos \theta^i}{n_1 \cos \theta^i + n_2 \cos \theta^i}; \quad R^{TE} = \frac{n_1 \cos \theta^i - n_2 \cos \theta^i}{n_1 \cos \theta^i + n_2 \cos \theta^i}. \tag{9.3.34}
\]
These expressions are called the Fresnel formulas. By using Snell’s law (9.1.20), we can rewrite them as (Exercise 9)

\[ T^{TM} = \frac{2 \sin \theta^i \cos \theta^i}{\sin(\theta^i + \theta^t)} \quad ; \quad R^{TM} = \frac{\tan(\theta^i - \theta^t)}{\tan(\theta^i + \theta^t)}, \]  

(9.3.35)

\[ T^{TE} = \frac{2 \sin \theta^i \cos \theta^i}{\sin(\theta^i + \theta^t)} \quad ; \quad R^{TE} = -\frac{\sin(\theta^i - \theta^t)}{\sin(\theta^i + \theta^t)}. \]  

(9.3.36)

At normal incidence where \( \theta^i = \theta^t = 0 \), we get from (9.3.33) and (9.3.34)

\[ T^{TE} = T^{TM} = \frac{2}{n + 1} \quad ; \quad R^{TM} = -R^{TE} = \frac{n - 1}{n + 1} \quad ; \quad n = \frac{n_2}{n_1}. \]  

(9.3.37)

The fact that \( R^{TM} = -R^{TE} \) at normal incidence follows from the way in which \( \hat{E}^{TM} \) and \( \hat{E}^{TE} \) are defined. From Fig. 9.4 we see that these two vectors point in opposite directions at normal incidence.

### 9.3.1 Reflectance and transmittance

Fig. 9.4 shows the polarisation vectors \( \hat{e}^{TMq} \) \((q = i, r, t)\) and \( \hat{e}^{TE} \) for TM and TE polarisation. These unit vectors are parallel with the electric field and follow from (9.3.9)-(9.3.12)

\[ \hat{e}^{TEi} = \hat{e}^{TER} = \hat{e}^{TEt} = \hat{e}^{TE} = \frac{k_i \times \hat{e}_z}{k_i} \quad ; \quad |\hat{e}^{TE}| = 1, \]  

(9.3.38)

\[ \hat{e}^{TMq} = \frac{k^q \times (k_i \times \hat{e}_z)}{k^q k_i} \quad ; \quad |\hat{e}^{TMq}| = 1. \]  

(9.3.39)

Let the angle between \( \mathbf{E}^q \) and the plane of incidence spanned by \( \mathbf{k}^q \) and \( \hat{e}^{TMq} \), be \( \alpha^q \) [see Fig. 9.5], so that

\[ \mathbf{E}^q = \hat{e}^{TE} E^q \sin \alpha^q + \hat{e}^{TMq} E^q \cos \alpha^q. \]  

(9.3.40)
Further, we let $J^i$, $J^r$, and $J^t$ denote the energy flows of respectively the incident, reflected, and transmitted fields per unit area of the interface. Then we have

$$J^{pq} = S^{pq} \cos \theta^q ; \quad p = TE, TM \quad ; \quad q = i, r, t,$$

(9.3.41)

where $S^{pq}$ is the absolute value of the Poynting vector, given by

$$S^{pq} = \frac{c}{4\pi} |E^{pq} \times H^{pq}| = \frac{c}{4\pi} E^{pq} H^{pq} = \frac{c}{4\pi} \sqrt{\varepsilon \mu} (E^{pq})^2 .$$

(9.3.42)

Here we have used the relation $\sqrt{\varepsilon \mu} E^{pq} = \sqrt{\varepsilon \mu} H^{pq}$. The reflectance $R^p (p = TE, TM)$ is the ratio between the reflected and incident energy flows. From (9.3.41)-(9.3.42) we have

$$R^{TM} = \frac{J^{TMr}}{J^{TMi}} = \frac{|E^{TMr}|^2}{|E^{TMi}|^2} = (R^{TM})^2 .$$

(9.3.43)

$$R^{TE} = \frac{J^{TEr}}{J^{TEi}} = \frac{|E^{TEr}|^2}{|E^{TEi}|^2} = (R^{TE})^2 .$$

(9.3.44)

Thus, the reflectance $R^p$ is equal to the square of reflection coefficient $R^p$.

The transmittance $T^p (p = TE, TM)$ is the ratio between the transmitted and incident energy flows, and (9.3.41)-(9.3.42) give

$$T^{TM} = \frac{J^{TMt}}{J^{TMi}} = \frac{n_2 \mu_1 \cos \theta^t}{n_1 \mu_2 \cos \theta^i} (T^{TM})^2 ,$$

(9.3.45)

$$T^{TE} = \frac{J^{TEt}}{J^{TEi}} = \frac{n_2 \mu_1 \cos \theta^t}{n_1 \mu_2 \cos \theta^i} (T^{TE})^2 .$$

(9.3.46)

Thus, the transmittance $T^p$ is proportional to the square of the transmission coefficient $T^p$ ($p = TE, TM$). When $\mu_2 = \mu_1 = 1$, we find on substitution from (9.3.35)-(9.3.36) into (9.3.43)-(9.3.46) the following expressions for the reflectance and the transmittance

$$R^{TM} = \tan^2(\theta^i - \theta^t),$$

(9.3.47)
\( R^{TE} = \frac{\sin^2(\theta^i - \theta^t)}{\sin^2(\theta^i + \theta^t)}, \)  
(9.3.48)

\( T^{TM} = \frac{\sin 2\theta^t \sin 2\theta^i}{\sin^2(\theta^i + \theta^t) \cos^2(\theta^i - \theta^t)}, \)  
(9.3.49)

\( T^{TE} = \frac{\sin 2\theta^t \sin 2\theta^i}{\sin^2(\theta^i + \theta^t)}. \)  
(9.3.50)

By use of these formulas one can show that

\[ R^{TM} + T^{TM} = 1; \quad R^{TE} + T^{TE} = 1, \]  
(9.3.51)

so that for each of the two polarisations the sum of the reflected energy and the transmitted energy is equal to the incident energy.

From (9.3.41) and (9.3.42) we have

\[ J^{pi} = \frac{c}{4\pi} \sqrt{\frac{\mu_1}{\epsilon_1}} |E^{pi}|^2 \cos \theta^i, \]  
(9.3.52)

which by the use of (9.3.40) gives

\[ J^{TEi} = \cos \theta^i \frac{c}{4\pi} \sqrt{\frac{\mu_1}{\epsilon_1}} |E^{TEi}|^2 = \cos \theta^i \frac{c}{4\pi} \sqrt{\frac{\mu_1}{\epsilon_1}} E^2 \sin^2 \alpha^i, \]  
(9.3.53)

\[ J^{TMi} = \cos \theta^i \frac{c}{4\pi} \sqrt{\frac{\mu_1}{\epsilon_1}} |E^{TMi}|^2 = \cos \theta^i \frac{c}{4\pi} \sqrt{\frac{\mu_1}{\epsilon_1}} E^2 \cos^2 \alpha^i. \]  
(9.3.54)

But since the total incident energy flow is given by

\[ J^i = \cos \theta^i \frac{c}{4\pi} \sqrt{\frac{\mu_1}{\epsilon_1}} E^2, \]  
(9.3.55)

we find

\[ J^{TEi} = J^i \sin^2 \alpha^i; \quad J^{TMi} = J^i \cos^2 \alpha^i. \]  
(9.3.56)

Thus, we have

\[ R = \frac{J^r}{J^i} = \frac{J^{TMr} + J^{TEr}}{J^i} = \frac{J^{TMr}}{J^{TMi}} \cos^2 \alpha^i + \frac{J^{TEr}}{J^{TEi}} \sin^2 \alpha^i, \]  
(9.3.57)

which gives

\[ R = R^{TM} \cos^2 \alpha^i + R^{TE} \sin^2 \alpha^i, \]  
(9.3.58)

and similarly we find

\[ T = T^{TM} \cos^2 \alpha^i + T^{TE} \sin^2 \alpha^i. \]  
(9.3.59)

At normal incidence, \( \theta^i = \theta^t = 0 \), and the distinction between \( TE \) and \( TM \) polarisation disappears. From (9.3.43)-(9.3.46) combined with (9.3.33)-(9.3.34), we find (when \( \mu_1 = \mu_2 = 1 \))

\[ R = R^{TM} = R^{TE} = (R^{TM})^2 = (R^{TE})^2 = \left( \frac{n - 1}{n + 1} \right)^2; \quad n = \frac{n_2}{n_1}, \]  
(9.3.60)

\[ T = T^{TM} = T^{TE} = (T^{TM})^2 = (T^{TE})^2 = \frac{4n}{(n + 1)^2}; \quad n = \frac{n_2}{n_1}. \]  
(9.3.61)

When \( n \to 1 \), we see that \( R \to 0 \) and \( T \to 1 \), as expected. Similarly, we find from (9.3.47)-(9.3.50) that \( R_{\parallel} \to 0, R_{\perp} \to 0, T_{\parallel} \to 1, T_{\perp} \to 1 \) when \( n \to 1 \).
9.3.2 Brewster’s law

From (9.3.47) it follows that \( R_{TM} = 0 \) when \( \theta_i + \theta_t = \frac{\pi}{2} \), since then \( \tan(\theta_i + \theta_t) = \infty \). We call this particular angle of incidence \( \theta_iB \) and the corresponding refraction or transmission angle \( \theta_tB \). By using Snell’s law (9.1.20), we find

\[
n_2 \sin \theta_tB = n_2 \sin \left( \frac{\pi}{2} - \theta_iB \right) = n_2 \cos \theta_iB = n_1 \sin \theta_iB,
\]

so that \( R_{TM} = 0 \) when \( \theta_i = \theta_iB \), where \( \theta_iB \) is given by

\[
\tan \theta_iB = \frac{n_2}{n_1} = n.
\]

The angle \( \theta_iB \) is called the polarisation angle or the Brewster angle. When the angle of incidence is equal to \( \theta_iB \), the \( E \) vector of the reflected light has no component in the plane of incidence (Fig. 9.6). This fact is exploited in sunglasses with polarisation filter. The filter is oriented such that only light that is polarised vertically (Fig. 9.6) is transmitted. Thus, one avoids to a certain degree annoying reflections from e.g. a water surface.

Note that \( k_r \cdot k_t = 0 \), i.e. \( k_r \) and \( k_t \) are normal to one another when \( \theta_i = \theta_iB \), as shown in Fig. 9.6.

9.3.3 Unpolarised light (natural light)

For natural light, e.g. light from an incandescent lamp, the direction of the \( E \) vector varies very rapidly in an arbitrary or irregular manner, so that no particular direction is given preference. The average reflectance \( \overline{R} \) is obtained by averaging over all directions \( \alpha \). Since the average value of both \( \sin^2 \alpha \) and \( \cos^2 \alpha \) is \( \frac{1}{2} \), we find from (9.3.56) that

\[
\overline{J_{TMi}} = J_i \overline{\cos^2 \alpha} = J_i \overline{\sin^2 \alpha} = \frac{1}{2} J_i.
\]

For the reflected components we find

\[
\overline{J_{TMr}} = \frac{J_{TMr}}{J_{TMi}} \overline{J_{TMi}} = \frac{J_{TMr}}{J_{TMi}} \cdot \frac{1}{2} J_i = \frac{1}{2} R_{TM} J_i,
\]
\[ J^{TEr} = J^{TEm} \cdot \frac{1}{2} J^i = \frac{1}{2} R^{TE} J^i, \]  

which shows that the degree of polarisation for the reflected light can be defined as
\[ P^r = \frac{|R^{TM} - R^{TE}|}{R^{TM} + R^{TE}} = \frac{|J^{TMr} - J^{TEm}|}{J^{TMr} + J^{TEm}}. \]

The average reflectance is given by
\[ R = \frac{\mathcal{R}}{J} = \frac{J^{TMr}}{J} + \frac{J^{TEm}}{J} = \frac{J^{TMr}}{2J^{TM}i} + \frac{J^{TEm}}{2J^{TE}i} = \frac{1}{2} (R^{TM} + R^{TE}), \]

so that the degree of polarisation becomes
\[ P^r = \frac{1}{R} \frac{1}{2} |R^{TM} - R^{TE}|, \]

where \(|R^{TM} - R^{TE}|\) is called the polarised part of the reflected light.

Similarly, we find for the transmitted light
\[ T = \frac{1}{2} (T^{TM} + T^{TE}); \quad P^t = \frac{1}{T} \frac{1}{2} |T^{TM} - T^{TE}|. \]

### 9.3.4 Rotation of the plane of polarisation upon reflection and refraction

Note that if the incident light is linearly polarised, then also the reflected and the transmitted light will be linearly polarised, since the phases only change by 0 or \(\pi\). This follows from the fact that the reflection and transmission coefficients are real quantities [cf. (9.3.33)-(9.3.36)]. But the planes of polarisation for the reflected and the transmitted light are rotated in opposite directions relative to the polarisation plane of the incident light. The angles \(\alpha^i, \alpha^r,\) and \(\alpha^t\) that the planes of polarisation of the incident, reflected, and transmitted light form with the plane of incidence, are given by [cf. Fig. 9.5]

\[ \tan \alpha^i = \frac{E^{TEi}}{E^{TMi}}, \]
\[ \tan \alpha^r = \frac{E^{TEr}}{E^{TMr}} = \frac{E^{TEr}}{E^{TMr}} \frac{E^{TEi}}{E^{TMi}} = \frac{R^{TE}}{R^{TM}} \tan \alpha^i, \]
\[ \tan \alpha^t = \frac{E^{TEE}}{E^{TMe}} = \frac{E^{TEE}}{E^{TMe}} \frac{E^{TEE}}{E^{TMt}} = \frac{T^{TE}}{T^{TM}} \tan \alpha^i. \]

By use of the Fresnel formulas (9.3.35)-(9.3.36) we can write
\[ \tan \alpha^r = -\frac{\cos(\theta^i - \theta^t)}{\cos(\theta^i + \theta^t)} \tan \alpha^i, \]
\[ \tan \alpha^t = \cos(\theta^t - \theta^i) \tan \alpha^i. \]

Since \(0 \leq \theta^i \leq \frac{\pi}{2}\) and \(0 \leq \theta^t \leq \frac{\pi}{2}\), we get
\[ |\tan \alpha^r| \geq |\tan \alpha^i|, \]
\[ |\tan \alpha^t| \leq |\tan \alpha^i|. \]

In (9.3.76) the equality sign applies at normal incidence \((\theta^i = \theta^t = 0)\) and at grazing incidence \((\theta^i = \frac{\pi}{2})\), whereas in (9.3.77) the equality sign applies only at normal incidence. These two inequalities
imply that upon reflection the plane of polarisation is rotated away from the plane of incidence, whereas upon transmission it is rotated towards the plane of incidence. Note that when $\theta^i = \theta^i_B$, so that $\theta^iB + \theta^tB = \frac{\pi}{2}$, then $\tan \alpha^r = \infty$. Thus, we have $\alpha^r = \frac{\pi}{2}$ in accordance with Brewster’s law.

### 9.3.5 Total reflection

Snell’s law (9.1.20) can be written in the form

$$\sin \theta^t = \frac{\sin \theta^i}{n} ; \quad n = \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2 \mu_2}{\varepsilon_1 \mu_1}}. \quad (9.3.78)$$

Hence, it follows that if $n < 1$, then we get $\sin \theta^t = 1$ when $\theta^i = \theta^i_c$, where

$$\sin \theta^i_c = n. \quad (9.3.79)$$

This implies that when $\theta^i = \theta^i_c$, we get $\theta^i = \theta^i_c$, so that the transmitted light propagates along the interface. If $\theta^i \geq \theta^i_c$, we have total reflection, i.e. no light will pass into the other medium. All light is then reflected. There exists a field in the other medium, but there is no energy transport through the interface. When $\theta^i > \theta^i_c$, then $\sin \theta^t > 1$, which means that $\theta^t$ is complex. We have from (9.3.78)

$$\cos \theta^t = \pm \sqrt{1 - \sin^2 \theta^t} = \pm i \sqrt{\frac{\sin^2 \theta^i}{n^2} - 1} = \frac{\pm i \sqrt{\sin^2 \theta^i - n^2}}{n}. \quad (9.3.80)$$

The lower sign in (9.3.80) must be discarded. Otherwise the field in medium 2 would grow exponentially with increasing distance from the interface. The electric field in medium 2 is

$$\mathbf{E}^{pt} = T^p \mathbf{E}^i e^{i(k^t \cdot r - \omega t)} \quad (p = TE, TM), \quad (9.3.81)$$

where

$$k^t \cdot r = k_x x + k_y y + k_z z, \quad (9.3.82)$$

with (cf. Fig. 9.1 and (9.3.80) with upper sign)

$$k_z = k_2 \cos \theta^t = \frac{k_2}{n} \sqrt{\sin^2 \theta^t - n^2} ; \quad n = \frac{n_2}{n_1} \quad (9.3.83)$$

so that

$$e^{ik^t \cdot r} = e^{i(k_x x + k_y y)} e^{-|k_z| z} ; \quad |k_z| = \frac{k_2}{n} \sqrt{\sin^2 \theta^t - n^2}. \quad (9.3.84)$$

We see that $\mathbf{E}^{pt}$ represents a wave that propagates along the interface and is exponentially damped with the distance $z$ into medium 2.

From $\nabla \cdot \mathbf{D}^t = \varepsilon_2 \nabla \cdot \mathbf{E}^t = 0$ it follows that

$$k^t \cdot \mathbf{E}^t = 0, \quad (9.3.85)$$

which gives

$$E^t_x = -\frac{k_x E^i_x + k_y E^i_y}{k_z}; \quad (9.3.86)$$

If we let the plane of incidence coincide with the $xz$ plane, we have (cf. Fig. 9.7)

$$k_x = -k_1 \sin \theta^i ; \quad k_y = 0, \quad (9.3.87)$$

$$E^t_y = E^T E^t e^{i(k_x x - \omega t)} e^{-|k_z| z} = T^T E^T E^t e^{i(k_x x - \omega t)} e^{-|k_z| z}, \quad (9.3.88)$$
Figure 9.7: Illustration of the refraction of a plane wave into an optically thinner medium, so that $\theta_i < \theta_t$. When $\theta_i \to \theta_i^c$, then $\theta_t \to \pi/2$, and we get total reflection.

\[
E_x^t = -E_{TMt}^T \cos \theta_t e^{i(k_x x - \omega t)} e^{-|k_z|z},
\]
\[
E_z^t = -\frac{k_x}{k_z^2} E_x^t = \frac{k_x}{k_z^2} T_{TMt} E_{TMi}^T e^{i(k_x x - \omega t)} e^{-|k_z|z}.
\]  

From these expressions for the components of $E^t$ and corresponding expressions for the components of $H^t$ one can show (Exercise 11) that the time average of the $z$ component of the Poynting vector is zero, which implies that there is no energy transport through the interface, as asserted earlier.

The reflection coefficients in (9.3.35)-(9.3.36) can be written as follows

\[
R_{TM} = \frac{\sin \theta_i \cos \theta_t - \sin \theta_t \cos \theta_i}{\sin \theta_i \cos \theta_t + \sin \theta_t \cos \theta_i},
\]
\[
R_{TE} = \frac{-\sin \theta_i \cos \theta_t - \sin \theta_t \cos \theta_i}{\sin \theta_i \cos \theta_t + \sin \theta_t \cos \theta_i}.
\]  

By combining Snell’s law (9.3.78) and (9.3.80) with the upper sign with (9.3.91)-(9.3.92), we get

\[
R_{TM} = \frac{n^2 \cos \theta_i - i \sqrt{n^2 \sin^2 \theta_i - 2}}{n^2 \cos \theta_i + i \sqrt{n^2 \sin^2 \theta_i - 2}},
\]
\[
R_{TE} = \frac{\cos \theta_i - i \sqrt{\sin^2 \theta_i - n^2}}{\cos \theta_i + i \sqrt{\sin^2 \theta_i - n^2}}.
\]  

Since both reflection coefficients are of the form $z/z^*$, where $z$ is a complex number, it follows that

\[
|R_{TM}| = |R_{TE}| = 1,
\]
which shows that for each polarisation the intensity of the totally reflected light is equal to the intensity of the incident light.

But the phase is altered upon total reflection. Letting

\[ R_{p} = \frac{E_{pr}}{E_{pi}} = e^{i\delta_{p}} = \frac{z_{p}}{z_{p}^*} = e^{2i\alpha_{p}} \quad (p = TE, TM), \]  

(9.3.96)

where [cf. (9.3.93)-(9.3.94)]

\[ z_{TM} = n^2 \cos \theta^i - i\sqrt{\sin^2 \theta^i - n^2} = |z_{TM}|e^{i\alpha_{TM}}, \]  

(9.3.97)

\[ z_{TE} = \cos \theta^i - i\sqrt{\sin^2 \theta^i - n^2} = |z_{TE}|e^{i\alpha_{TE}}, \]  

(9.3.98)

we find

\[ \tan \alpha_{TM} = \tan \left( \frac{1}{2} \delta_{TM} \right) = -\frac{\sqrt{\sin^2 \theta^i - n^2}}{n^2 \cos \theta^i}, \]  

(9.3.99)

\[ \tan \alpha_{TE} = \tan \left( \frac{1}{2} \delta_{TE} \right) = -\frac{\sqrt{\sin^2 \theta^i - n^2}}{\cos \theta^i}. \]  

(9.3.100)

The relative phase difference

\[ \delta = \delta_{TE} - \delta_{TM}, \]  

(9.3.101)

is determined by

\[ \tan \left( \frac{1}{2} \delta \right) = \tan \left( \frac{1}{2} \delta_{TE} \right) - \tan \left( \frac{1}{2} \delta_{TM} \right), \]  

(9.3.102)

which upon substitution from (9.3.99)-(9.3.100) gives

\[ \tan \left( \frac{1}{2} \delta \right) = \frac{\cos \theta^i \sqrt{\sin^2 \theta^i - n^2}}{\sin^2 \theta^i}. \]  

(9.3.103)

We see that \( \delta = 0 \) for \( \theta^i = \frac{\pi}{2} \) (grazing incidence) and \( \theta^i = \theta^{ic} \) (critical angle of incidence). Between these two values there is an angle of incidence \( \theta^i = \theta^{im} \) which gives a maximum phase difference \( \delta = \delta^m \), where \( \theta^{im} \) is determined by

\[ \frac{d\delta}{d\theta^i} \bigg|_{\theta^{im}} = 0. \]  

(9.3.104)

From (9.3.104) we find

\[ \sin^2 \theta^{im} = \frac{2n^2}{1 + n^2}, \]  

(9.3.105)

which upon substitution in (9.3.103) gives (Exercise 10)

\[ \tan \left( \frac{1}{2} \delta^m \right) = \frac{1 - n^2}{2n}. \]  

(9.3.106)

If the phase difference \( \delta \) is equal to \( \pm \frac{\pi}{2} \) and in addition \( E_{TMi} = E_{TEi} \), the totally reflected light will be circularly polarised. By choosing the angle \( \alpha^i \) between the polarisation plane and the plane of incidence equal to 45°, we make \( E_{TMi} \) equal to \( E_{TEi} \). In order to obtain \( \delta = \frac{\pi}{2} \) we must have \( \delta^m \geq \frac{\pi}{4} \). This means that \( \tan \left( \frac{\delta^m}{2} \right) \geq \tan \left( \frac{\pi}{4} \right) = 1 \), which according to (9.3.106) implies that

\[ n^2 + 2n - 1 \leq 0. \]  

(9.3.107)

By completing the square on the left-hand side of (9.3.107), we find that
\[ n \leq \sqrt{2} - 1 \; ; \; \frac{1}{n} = \frac{n_1}{n_2} \geq \sqrt{2} + 1 = 2.41. \] 

(9.3.108)

Thus, \( \frac{n_1}{n_2} \) must exceed 2.41 in order that we shall obtain a phase difference of \( \frac{\pi}{2} \) in one single reflection.