1 Maxwell’s Equations in Gaussian Units

\[ \nabla \times \mathbf{H} = \frac{1}{c} \dot{\mathbf{D}} + \frac{4\pi}{c} \mathbf{J} \]  
\[ \nabla \times \mathbf{E} = -\frac{1}{c} \dot{\mathbf{B}} \]  
\[ \nabla \cdot \mathbf{D} = 4\pi \rho \]  
\[ \nabla \cdot \mathbf{B} = 0. \]

1.1 Continuity Equation

\[ \nabla \cdot \mathbf{J} + \dot{\rho} = 0. \]  

- Integration over a closed volume \( V \) with surface \( S \), and use of the divergence theorem gives

\[ \oint_S \mathbf{J} \cdot \hat{n} da = -\frac{d}{dt} \iiint_V \rho dv = -\frac{d}{dt} Q. \]

- Here \( \hat{n} \) is the outward surface normal. Thus, the integrated current flux out of the closed volume \( V \) is equal to the loss of charge in the same volume.

1.2 Material Equations

- If the field is time harmonic and the matter is isotropic and at rest, the material equations are

\[ \mathbf{J}_c = \sigma \mathbf{E} ; \quad \mathbf{D} = \varepsilon \mathbf{E} ; \quad \mathbf{B} = \mu \mathbf{H}. \]  

- The total current density \( \mathbf{J} \) in (1) can in addition consist of an externally applied conductivity density \( \mathbf{J}_0 \), i.e.

\[ \mathbf{J} = \mathbf{J}_0 + \mathbf{J}_c = \mathbf{J}_0 + \sigma \mathbf{E}. \]

1.3 Boundary Conditions

- From Maxwell’s equations, combined with Stokes’ and Gauss’ theorems, one can derive the following boundary conditions

\[ \hat{n} \cdot (\mathbf{B}^{(2)} - \mathbf{B}^{(1)}) = 0 \]  
\[ \hat{n} \cdot (\mathbf{D}^{(2)} - \mathbf{D}^{(1)}) = 4\pi \rho_s \]  
\[ \hat{n} \times (\mathbf{E}^{(2)} - \mathbf{E}^{(1)}) = 0 \]  
\[ \hat{n} \times (\mathbf{H}^{(2)} - \mathbf{H}^{(1)}) = \frac{4\pi}{c} \mathbf{J}_s \]

where \( \rho_s \) is a possible surface charge density at the boundary, and \( \mathbf{J}_s \) is possible a surface current density at the boundary.

2 Poynting’s Vector and Energy Law

- Electric energy density \( w_e \):

\[ w_e = \frac{1}{8\pi} \mathbf{E} \cdot \mathbf{D}. \]

- Magnetic energy density \( w_m \):

\[ w_m = \frac{1}{8\pi} \mathbf{H} \cdot \mathbf{B}. \]
• Total energy density:

\[ w = w_e + w_m. \] (15)

• Poynting’s vector \( \mathbf{S} \), given by

\[ \mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \] (16)

represents the amount of energy that per unit time crosses a unit area that is parallel to both \( \mathbf{E} \) and \( \mathbf{H} \).

• Conservation law in a non-conducting medium (\( \sigma = 0 \)):

\[ \frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} = 0. \] (17)

• The absolute value of Poynting’s vector is proportional to the light intensity:

\[ |\mathbf{S}| \propto \text{light intensity}. \] (18)

• The unit vector \( \hat{s} \) along Poynting’s vector

\[ \hat{s} = \frac{\mathbf{S}}{|\mathbf{S}|} \] (19)

points in the direction of light propagation.

3 Wave Equation and Speed of Light

• We start with Maxwell’s curl equations:

\[ \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c} \varepsilon \mathbf{E} \] (20)

\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{c} \mu \mathbf{H} \] (21)

and assume that \( \varepsilon \) and \( \mu \) do not vary with position.

• Taking the curl of (21) and combining the result with the time derivative of (20), we find:

\[ \nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c} \mu \nabla \times \mathbf{H} = -\frac{1}{c} \frac{\partial}{\partial \mathbf{t}} \left( \varepsilon \mathbf{E} \right) = -\frac{\varepsilon \mu}{c^2} \mathbf{E}. \] (22)

• Now we use the vector relation

\[ \nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}, \] (23)

and the fact that \( \nabla \cdot \mathbf{E} = 0 \) to obtain

\[ \nabla^2 \mathbf{E} - \frac{\varepsilon \mu}{c^2} \mathbf{E} = 0. \] (24)

• Similarly, we find

\[ \nabla^2 \mathbf{H} - \frac{\varepsilon \mu}{c^2} \mathbf{H} = 0. \] (25)

• Comparison with the scalar wave equation

\[ \nabla^2 V - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} V = 0 \] (26)

shows that each Cartesian component of \( \mathbf{E} \) and \( \mathbf{H} \) satisfies the scalar wave equation with phase velocity

\[ v = \frac{c}{\sqrt{\varepsilon \mu}}. \] (27)

• Note: This derivation is restricted to non-dispersive media in which both the permittivity and the permeability do not depend on the frequency.

4 Scalar Waves

• Scalar waves are solutions of the scalar wave equation (26), which is

\[ \nabla^2 V(r, t) - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} V(r, t) = 0. \] (28)

4.1 Plane Waves

• Any solution of (28) in the form

\[ V(r, t) = V(r \cdot \hat{s}, t) \] (29)

is called a plane wave, since \( V \) at any time \( t \) is constant over any plane
Figure 2: A plane wave that propagates in direction \( \hat{s} \) has no variation in any plane that is normal to \( \hat{s} \).

\[ \mathbf{r} \cdot \hat{s} = \text{constant} \tag{30} \]

which is normal to the unit vector \( \hat{s} \) (see Fig. 2).

- We introduce a new variable

\[ \zeta = \mathbf{r} \cdot \hat{s} = xs_x + ys_y + zs_z \tag{31} \]

so that the wave equation becomes

\[ \frac{\partial^2 V}{\partial \zeta^2} - \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2} = 0. \tag{32} \]

- Next, we introduce two new variables \( p \) and \( q \)

\[ p = \zeta - vt ; \quad q = \zeta + vt \tag{33} \]

to obtain the wave equation in \( (p, q) \) variables as

\[ \frac{\partial^2 V}{\partial p \partial q} = 0. \tag{34} \]

- This equation has the following general plane-wave solution

\[ V(\mathbf{r}, t) = V_1(\mathbf{r} \cdot \hat{s} - vt) + V_2(\mathbf{r} \cdot \hat{s} + vt). \tag{35} \]

- Since

\[ \zeta - vt = \zeta + v\tau - v(t + \tau), \tag{36} \]

we have

\[ V_1(\zeta, t) = V_1(\zeta + v\tau, t + \tau). \tag{37} \]

- **CONCLUSION:** \( V(\zeta \pm vt) \) represents a plane wave that propagates at velocity \( v \) in the positive \( \zeta \) direction (upper sign) or in the negative \( \zeta \) direction (lower sign).

### 4.2 Spherical Waves

- Consider now solutions of the scalar wave equation (28) in the form

\[ V = V(\mathbf{r}, t) \tag{38} \]

where

\[ r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \tag{39} \]

is the distance from the origin \((0,0,0)\).

- We substitute

\[ \nabla^2 V = \frac{1}{r} \frac{\partial^2 V}{\partial r^2}(rV) \tag{40} \]

into the wave equation (28) to obtain

\[ \frac{\partial^2 V}{\partial r^2}(rV) - \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2}(rV) = 0. \tag{41} \]

- Comparison of (41) with (32) shows that (cfr. (35))

\[ rV = V_1(r - vt) + V_2(r + vt). \tag{42} \]

- **CONCLUSION:** \( V(r \mp vt) \) represents a diverging spherical wave that propagates away from the origin (upper sign) or a converging spherical wave that propagates towards the origin (lower sign).
4.3 Harmonic (Monochromatic) Waves

- At a given observation point \( r \) the solution of the wave equation is a function only of time, i.e.

\[
V(r, t) = F(t). \tag{43}
\]

- If \( F(t) \) has the simple form

\[
F(t) = a \cos(\omega t + \delta), \tag{44}
\]

we have a harmonic wave in time. Here

- \( a \) is the amplitude (positive),
- \( \omega \) is the angular frequency, and
- \( \omega t + \delta \) is the phase.

- A harmonic wave is called monochromatic because it has only one frequency or wavelength component.

- The frequency \( \nu \) and the period \( T \) follow from

\[
\nu = \frac{\omega}{2\pi} = \frac{1}{T}. \tag{45}
\]

- The harmonic wave in (44) has period \( T \) because

\[
F(t + T) = a \cos(\omega(t + T) + \delta) = a \cos(\omega t + \delta + 2\pi) = F(t). \tag{46}
\]

- The general expression for a plane wave propagating in the \( \hat{s} \) direction can be written

\[
V = V_1(r \cdot \hat{s} - vt) = V_1' \left[ \frac{\nu}{\omega} \left( \frac{\omega}{v} r \cdot \hat{s} - \omega t \right) \right] = V_1' (k \cdot r - \omega t), \tag{47}
\]

where \( k = k\hat{s} = \frac{\omega}{v} \hat{s} \), and where \( V_1 \) and \( V_1' \) both are arbitrary functions.

- By replacing \( \omega t \) in (44) by \( k \cdot r - \omega t \), we get a harmonic plane wave:

\[
V(r, t) = a \cos(k \cdot r - \omega t - \delta). \tag{48}
\]

- Note that (48) remains unchanged if we replace \( r \cdot \hat{s} \) with \( r \cdot \hat{s} + n\lambda \), where \( n = 1, 2, \ldots \), and \( \lambda \) is given by

\[
\lambda = \frac{2\pi}{k} = \nu \frac{2\pi}{\omega} = vT = \frac{v}{\nu}. \tag{49}
\]

- The quantity \( \lambda \) is called the wavelength. For \( t = \text{constant} \), \( V(r, t) \) in (48) is periodic with wavelength \( \lambda \):

\[
V(r \cdot \hat{s}, t) = V(r \cdot \hat{s} + n\lambda, t) ; \quad n = 1, 2, 3, \ldots . \tag{50}
\]

- Wave number

\[
k = \frac{2\pi}{\lambda} = \frac{\omega}{v}. \tag{51}
\]

- Wave vector

\[
k = k\hat{s}. \tag{52}
\]

- Plane, harmonic wave:

\[
V(r, t) = a \cos(k \cdot r - \omega t - \delta). \tag{53}
\]

- Converging or diverging harmonic spherical wave:

\[
V(r, t) = a \cos(\mp kr - \omega t - \delta). \tag{54}
\]

- The upper sign corresponds to a converging spherical wave and the lower sign to a diverging spherical wave.

- Plane, harmonic wave propagating in the positive \( z \) direction:

\[
V(z, t) = a \cos(kz - \omega t - \delta). \tag{55}
\]
• The phase is constant on a wave front:

\[ \phi = kz - \omega t - \delta = \text{constant}. \]  

(56)

• Thus, on a wave front:

\[ z = vt + \text{constant} ; \quad v = \frac{\omega}{k}. \]  

(57)

• CONCLUSION: A wave front propagates at the phase velocity given by:

\[ v = \frac{\omega}{k}. \]  

(58)

4.4 Complex Representation

\[ V(r, t) = \text{Re}\{U(r)e^{-i\omega t}\}, \]  

(59)

• Plane Wave:

\[ U(r) = ae^{i(kr - \delta)}. \]  

(60)

• Diverging (upper sign) or Converging (lower sign) Spherical Wave:

\[ U(r) = \frac{a}{r}e^{i(\pm kr - \delta)}. \]  

(61)

• When performing linear operations, such as differentiation, integration or summation, we can drop the Re symbol during the operations, as long as we remember to take the real part of the result in the end.

• Substituting

\[ V(r, t) = U(r)e^{-i\omega t} \]  

(62)

into the wave equation (28), we find that the complex amplitude \( U(r) \) satisfies the Helmholtz equation:

\[(\nabla^2 + k^2)U(r) = 0.\]  

(63)

4.5 Linearity and Superposition Principle

• For any linear equation the sum of two or several solutions is also a solution. This is called the superposition principle.

• Since Maxwell’s equations are linear, the superposition principle is valid for electromagnetic waves as long as the material equations are linear.

4.6 Phase Velocity and Group Velocity

• A general wave \( V(r, t) \) can always be expressed as a sum of harmonic components.

• If \( \epsilon \) depends on \( \omega \), i.e. \( \epsilon = \epsilon(\omega) \), the phase velocity also will depend on \( \omega \), since \( v = \frac{\omega}{k} = v(\omega) \).

• Thus, different harmonic components will have different phase velocities.

• A polychromatic wave or a pulse comprised of many harmonic components therefore will change its form during propagation.

• The energy will not propagate at the phase velocity, but at the group velocity, defined as

\[ v_g = \frac{d\omega}{dk}. \]  

(64)

• If \( n(\omega) = \text{constant} \), we have a non-dispersive medium. Since \( \omega = vk \), and the phase velocity \( v = \frac{\omega}{k} \) now is constant, we have

\[ v_g = \frac{d}{dk}(vk) = v. \]  

• Thus, the phase velocity and the group velocity are equal in a non-dispersive medium where \( n = \text{constant} \).

• In dispersive media we have

\[ v_g = \frac{d}{dk}(vk) = v + \frac{dv}{dk} = v - \lambda \frac{dv}{d\lambda}. \]  

(65)
5 Repetition

- From Maxwell’s equations in a source-free, non-dispersive medium \((\mathbf{J} = 0; \rho = 0)\) we have
  \[ \nabla^2 \mathbf{E} - \frac{\varepsilon \mu}{c^2} \ddot{\mathbf{E}} = 0; \quad \nabla^2 \mathbf{H} - \frac{\varepsilon \mu}{c^2} \ddot{\mathbf{H}} = 0. \]  
  \(\text{(66)}\)

- Comparison of (66) with the scalar wave equation
  \[ \nabla^2 V - \frac{1}{v^2} \dot{V} = 0 \]  
  \(\text{(67)}\)

shows that each Cartesian component of \(\mathbf{E}\) and \(\mathbf{H}\) satisfies the scalar wave equation with phase velocity \(v\) given by
  \[ v = \frac{c}{\sqrt{\varepsilon \mu}} = \frac{c}{n}. \]  
  \(\text{(68)}\)

- The scalar wave equation (67) has elementary solutions in the form of plane waves or spherical waves.

Plane Waves

- For a plane wave \(V\) is given by
  \[ V(\mathbf{r}, t) = V_1(\mathbf{r} \cdot \hat{s} - vt) + V_2(\mathbf{r} \cdot \hat{s} + vt) \]  
  \(\text{(69)}\)

where \(V(\zeta \mp vt)\) represents a plane wave that propagates in the positive \(\zeta\) direction (upper sign) or in the negative \(\zeta\) direction (lower sign).

Spherical Waves

- For a spherical wave \(V\) is given by
  \[ V(\mathbf{r}, t) = \frac{V_1(\mathbf{r} - vt)}{r} + \frac{V_2(\mathbf{r} + vt)}{r} \]  
  \(\text{(70)}\)

where \(\frac{V(\mathbf{r} \mp vt)}{r}\) represents a spherical wave that propagates away from the origin (upper sign) or towards the origin (lower sign).

Harmonic (Monochromatic) Waves

- A plane harmonic wave that propagates in the direction \(\mathbf{k} = k\hat{s}\) is given by
  \[ V(\mathbf{r}, t) = a \cos(k \cdot \mathbf{r} - \omega t + \delta). \]  
  \(\text{(71)}\)

- The corresponding spherical wave is
  \[ V(\mathbf{r}, t) = \frac{a}{r} \cos(\pm kr - \omega t + \delta) \]  
  \(\text{(72)}\)

where the upper sign represents a diverging spherical wave and the lower sign represents a converging spherical wave.

Complex representation of harmonic waves

- In complex notation we have
  \[ V(\mathbf{r}, t) = \text{Re}[U(\mathbf{r})e^{-i\omega t}]. \]  
  \(\text{(73)}\)

- For a plane wave the complex amplitude \(U(\mathbf{r})\) is given by
  \[ U(\mathbf{r}) = ae^{i(k \cdot \mathbf{r} + \delta)} \]

and for a diverging or converging spherical wave:
  \[ U(\mathbf{r}) = \frac{a}{r} e^{i(\pm kr + \delta)}. \]

- Substituting (73) into the wave equation (67), we find that \(U(\mathbf{r})\) satisfies the Helmholtz equation, i.e.
  \[ (\nabla^2 + k^2)U(\mathbf{r}) = 0. \]  
  \(\text{(74)}\)

6 Pulse Propagation in a Dispersive Medium

- Consider a polychromatic, plane wave that propagates in the positive \(z\) direction in a linear, homogeneous, isotropic, and dispersive medium that fills the half space \(z > 0\) (Fig. 3).
The plane wave \( u(z, t) \) is comprised of different harmonic components, implying that we can represent \( u(z, t) \) by the following inverse Fourier transform

\[
u(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(z, \omega)e^{-i\omega t} d\omega \tag{75}\]

where the frequency spectrum \( \tilde{u}(z, \omega) \) is given as the Fourier transform of \( u(z, t) \), i.e.

\[
\tilde{u}(z, \omega) = \int_{-\infty}^{\infty} u(z, t)e^{i\omega t} dt. \tag{76}\]

The solution is:

\[
u(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega)e^{i(k(\omega)z - \omega t)} d\omega \tag{77}\]

or

\[
u(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega)e^{i\frac{v\omega}{c}f(\omega)} d\omega, \tag{78}\]

where \( \tilde{g}(\omega) \) is the frequency spectrum of the pulse in the plane \( z = 0 \), and where

\[
f(\omega) = \omega[n(\omega) - \theta] ; \quad \theta = \frac{ct}{z}. \tag{79}\]

For the special case in which \( n(\omega) = \frac{\omega}{v(\omega)} = \text{constant} \), which implies that we have a non-dispersive medium, we obtain

\[
u(z, t) = u \left(0, -\frac{z}{v} + t\right). \tag{80}\]

This result shows that in a non-dispersive medium the pulse propagates in the positive \( z \) direction at velocity \( v \) without changing its form.

What happens to the propagation of a pulse with a very narrow frequency spectrum?

**Lorentz medium**

A model commonly used to study pulse propagation in dispersive media is the Lorentz medium with one single resonance frequency. Then the refractive index \( n(\omega) \) is given by

\[
n(\omega) = \left[1 - \frac{b^2}{\omega^2 - \omega_0^2 + 2\delta i\omega}\right]^{1/2} \tag{81}\]

where \( b \) is a constant, \( \omega_0 \) is the resonance frequency, and \( \delta \) represents the damping (attenuation) in the medium.

Equation (77) shows that in a dispersive medium:

\( u(z, t) \) (for any \( z > 0 \)) is a sum of harmonic plane waves of the form

\[
g(\omega)\exp[i(k(\omega)z - \omega t)] = \tilde{g}(\omega)\exp[-ik_1(\omega)z]\exp[i(k_r(\omega)z - \omega t)]. \]

\( k_r(\omega) \) and \( k_i(\omega) \) are the real and the imaginary part, respectively, of \( k(\omega) \).

The amplitude \( \tilde{g}(\omega)\exp[-ik_1(\omega)z] \), is damped exponentially as \( z \) increases.

The phase velocity is given by \( v(\omega) = \frac{c}{k_r(\omega)} \), where \( k(\omega) = (\omega/c)n(\omega) = (\omega/c)[n_r(\omega) + in_i(\omega)] \).

Since the phase velocity \( v \) depends on the frequency \( \omega \), the different plane waves will arrive at a given position \( z \) at different times and thus cause a distortion of the pulse, i.e. the form of the pulse will change.

Also, the damping factor \( k_i(\omega) \) depends on \( \omega \), so that the different frequency components will have different amplitudes when they arrive at a given position \( z \).
7 General Electromagnetic Plane Wave

- General electromagnetic plane wave:
  \[ \mathbf{E} = 
  \mathbf{E}(\mathbf{k} \cdot \mathbf{r} - \omega t) \;
  \mathbf{H} = 
  \mathbf{H}(\mathbf{k} \cdot \mathbf{r} - \omega t). \]  
  \[ (82) \]

- Here \( \mathbf{k} = k \hat{s} \) with \( \hat{s} \) pointing in the direction of propagation.

- Writing \( u = \mathbf{k} \cdot \mathbf{r} - \omega t \) and using Maxwell’s curl equations in source-free space, we obtain
  \[ \mathbf{k} \times \mathbf{H}' = \frac{\varepsilon}{c}(-\omega)\mathbf{E}' \]  
  \[ (83) \]
  
  \[ \mathbf{k} \times \mathbf{E}' = -\frac{\mu}{c}(-\omega)\mathbf{H}' \]  
  \[ (84) \]
  where the prime denotes differentiation with respect to \( u \).

- Integrating over \( u \) and setting the integration constant equal to zero, we get
  \[ \mathbf{E} = -\frac{1}{k_0 \varepsilon} \mathbf{k} \times \mathbf{H} \;
  \mathbf{H} = \frac{1}{k_0 \mu} \mathbf{k} \times \mathbf{E}'. \]  
  \[ (85) \]

- Scalar multiplication with \( \mathbf{k} \) gives
  \[ \mathbf{k} \cdot \mathbf{E} = \mathbf{k} \cdot \mathbf{H} = 0. \]  
  \[ (86) \]

- Thus, both \( \mathbf{E} \) and \( \mathbf{H} \) are transverse waves, and the vectors \( \hat{s}, \mathbf{E}, \) and \( \mathbf{H} \) form a right-handed Cartesian co-ordinate system.

- Electric and magnetic energy densities:
  \[ w_e = \frac{1}{8\pi} \mathbf{E} \cdot \mathbf{D} = \frac{\varepsilon}{8\pi} E^2 \;
  E = |\mathbf{E}|, \]  
  \[ (87) \]

  \[ w_m = \frac{1}{8\pi} \mathbf{B} \cdot \mathbf{H} = \frac{\mu}{8\pi} H^2 \;
  H = |\mathbf{H}|. \]  
  \[ (88) \]

- Since \( \sqrt{\mu H} = \sqrt{\varepsilon E} \), we have \( w_e = w_m \), and hence:

- Total energy density:
  \[ w = w_e + w_m = 2w_e = \frac{1}{4\pi} \varepsilon E^2 = 2w_m = \frac{1}{4\pi} \mu H^2, \]  
  \[ (89) \]

- Poynting vector:
  \[ \mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} = \frac{c}{4\pi} \mathbf{E} \mathbf{H} \hat{s} = \frac{c}{4\pi} E \sqrt{\frac{\varepsilon}{\mu}} \mathbf{E} \hat{s} = \left( \frac{1}{4\pi} \varepsilon E^2 \right) \left( \frac{c}{\sqrt{\varepsilon \mu}} \right) \hat{s} = \mathbf{w} \mathbf{s}. \]  
  \[ (90) \]

- Dimensional analysis:
  \[ [\mathbf{S}] = [w][v] = \frac{\text{Energy}}{\text{m}^3 \cdot \text{s}} \cdot \frac{\text{m}}{\text{s}} = \frac{\text{Energy}}{\text{m}^2 \cdot \text{s}} \]  
  \[ (91) \]

- Thus, \( \mathbf{S} \) represents the amount of energy per unit time that passes through a unit area of the plane that is spanned by \( \mathbf{E} \) and \( \mathbf{H} \).
8 Harmonic Waves of Arbitrary Form - Time Averages

- Harmonic electromagnetic wave of arbitrary form:
  \[ \mathbf{E} = \text{Re}\left\{ \mathbf{E}_0 r e^{-i\omega t} \right\} \quad ; \quad \mathbf{H} = \text{Re}\left\{ \mathbf{H}_0 r e^{-i\omega t} \right\} . \]  
  \(92\)

- Since optical frequencies are very high, we can only observe the average of \(w_e, w_m\) or \(S\), taken over a time interval \(-T' \leq t \leq T'\), where \(T'\) is much larger than the period \(T = \frac{2\pi}{\omega}\).

- For the time average of the electric energy density we have
  \[ \langle w_e \rangle = \frac{1}{2T'} \int_{-T'}^{T'} \frac{c}{8\pi} |\mathbf{E}|^2 dt . \]  
  \(93\)

- For any complex number \(z\), we have \(\text{Re} z = \frac{1}{2}(z + z^*)\), where \(z^*\) is the complex conjugate of \(z\). Thus:
  \[ \mathbf{E} = \text{Re}[\mathbf{E}_0 r e^{-i\omega t}] = \frac{1}{2}[\mathbf{E}_0 e^{-i\omega t} + \mathbf{E}_0^* e^{i\omega t}] \]
  
  \[|\mathbf{E}|^2 = \mathbf{E} \cdot \mathbf{E}^* = \frac{1}{4} [\mathbf{E}_0^2 e^{-2i\omega t} + 2\mathbf{E}_0 \cdot \mathbf{E}_0^* + \mathbf{E}_0^* e^{2i\omega t}] . \]  
  \(94\)

- Further:
  \[ \frac{1}{2T'} \int_{-T'}^{T'} \exp(\pm 2i\omega t) dt = \frac{1}{4\pi T} \frac{T}{T'} \sin(2\omega T') . \]  
  \(95\)

- Since \(T' \gg T\), integrals that include a factor \(\exp(\pm 2i\omega t)\) can be neglected, and hence
  \[ \langle w_e \rangle = \frac{c}{16\pi} \mathbf{E}_0 \cdot \mathbf{E}_0^* . \]  
  \(96\)

- Similarly, we obtain:
  \[ \langle w_m \rangle = \frac{\mu}{16\pi} \mathbf{H}_0 \cdot \mathbf{H}_0^* . \]  
  \(97\)

- The time average of the Poynting vector is:
  \[ \langle \mathbf{S} \rangle = \frac{1}{2T'} \int_{-T'}^{T'} \frac{c}{4\pi} (\mathbf{E} \times \mathbf{H}) dt \]  
  where
  \[ \mathbf{E} \times \mathbf{H} = \frac{1}{4} \{ \mathbf{E}_0 \times \mathbf{H}_0 e^{-2i\omega t} + \mathbf{E}_0 \times \mathbf{H}_0^* + \mathbf{E}_0^* \times \mathbf{H}_0 + \mathbf{E}_0^* \times \mathbf{H}_0^* e^{2i\omega t} \} . \]  
  \(98\)

- Hence
  \[ \langle \mathbf{S} \rangle = \frac{c}{16\pi} \{ \mathbf{E}_0 \times \mathbf{H}_0^* + \mathbf{E}_0^* \times \mathbf{H}_0 \} = \frac{c}{8\pi} \text{Re}(\mathbf{E}_0 \times \mathbf{H}_0^*) . \]  
  \(99\)

9 Harmonic Plane Wave – Polarisation

- Each Cartesian component of \(\mathbf{E}\) and \(\mathbf{H}\) is of the form
  \[ a \cos(\tau + \delta) = \quad ; \quad a > 0 \quad ; \quad \tau = k \cdot r - \omega t . \]  
  \(100\)

- If \(k\) points along the \(z\) axis, then only the \(x\) and \(y\) components of \(\mathbf{E}\) and \(\mathbf{H}\) are non-zero. The end point of the electric vector describes a curve having co-ordinates \((E_x, E_y)\) given by
  \[ E_x = a_1 \cos(\tau + \delta_1) \quad ; \quad a_1 > 0 \]  
  \(102\)

  \[ E_y = a_2 \cos(\tau + \delta_2) \quad ; \quad a_2 > 0 . \]  
  \(103\)

- Eliminating \(\tau\) from (102)-(103), we obtain the equation of an ellipse:
  \[ \left( \frac{E_x}{a_1} \right)^2 + \left( \frac{E_y}{a_2} \right)^2 - 2 \frac{E_x}{a_1} \frac{E_y}{a_2} \cos \delta = \sin^2 \delta \]  
  \(104\)

  where \(\delta = \delta_2 - \delta_1\).

- The cross term implies that the ellipse is rotated relative to the \((x, y)\) coordinate axes.
• For $\delta = \frac{\pi}{2}$, the cross term vanishes, and we have

$$\left( \frac{E_x}{a_1} \right)^2 + \left( \frac{E_y}{a_2} \right)^2 = 1. \quad (105)$$

• In a $(\xi, \eta)$ co-ordinate system coinciding with the axes of the ellipse, we have

$$E_\xi = a \cos(\tau + \delta_0) \quad ; \quad E_\eta = \pm b \sin(\tau + \delta_0) \quad (106)$$

which gives

$$\left( \frac{E_\xi}{a} \right)^2 + \left( \frac{E_\eta}{b} \right)^2 = 1. \quad (107)$$

• When the upper or lower sign in (106) applies, the electric vector rotates against or with the clock, respectively, if we view the $xy$ plane from the positive $z$ axis.

• Rotation against the clock is called left-handed polarisation, and rotation with the clock is called right-handed polarisation.

**Linear polarisation.** If $\delta = m\pi \quad (m = 1, \pm 1, \pm 2, \ldots)$, then the ellipse in (102)-(103) degenerates into a straight line, i.e.

$$\frac{E_y}{E_x} = (-1)^m \frac{a_2}{a_1}. \quad (108)$$

**Circular polarisation.** If $a_1 = a_2$ and $\delta = \pm \frac{\pi}{2} + 2m\pi \quad m = (0, \pm 1, \pm 2, \ldots)$, then the ellipse in (102)-(103) degenerates into a circle, i.e. (with $\beta = \tau + \delta_1$)

$$E_x = a \cos \beta \quad ; \quad E_y = a \cos \left( \beta + 2m\pi \pm \frac{\pi}{2} \right) = \mp a \sin \beta \quad (109)$$

which gives

$$E_x^2 + E_y^2 = a^2. \quad (110)$$

• We have right-handed circular polarisation when $E_y = -a \sin \beta$ and left-handed circular polarisation when $E_y = +a \sin \beta$.

## 10 Reflection and refraction of a plane wave

• Let a plane wave be incident upon a plane interface between two different media (Fig. 7), giving rise to a reflected plane wave and a transmitted plane wave.

• Each component of $E$ or $H$ can be written

$$A_j = \text{Re}\{a_j e^{i(k_j r - \omega t)}\} \quad (j = x, y, z) \quad (111)$$

where $A$ stands for $E$ or $H$ and $q = i, r, t$. 

Figure 5: Instantaneous picture of the electric vector of a plane wave that propagates in the $z$ direction.

Figure 6: The end point of the electric vector describes an ellipse that is inscribed in a rectangle with sides $2a_1$ and $2a_2$. 
Figure 7: Reflection of a plane wave at a plane interface between two different media. Illustration of propagation directions and angles of incidence, reflection, and transmission.

Figure 8: Reflection and refraction of a plane wave. Illustration of the co-ordinate system (\(\hat{n}, \hat{b}, \hat{t}\)).

- Thus, \(k^i\), \(k^r\), and \(k^t\) are the wave vectors of the incident, reflected, and transmitted wave, respectively.

10.1 Reflection law and refraction law (Snell’s law)

- The existence of continuity conditions that \(E\) and \(H\) must satisfy, gives

\[
k^i \cdot r = k^r \cdot r = k^t \cdot r. \tag{112}\]

- We introduce a Cartesian co-ordinate system with unit vectors \(\hat{n}, \hat{b}, \hat{t}\) (Fig. 8), with \(\hat{n}\) pointing along the interface normal into the medium of the refracted wave, and with \(\hat{b}\) and \(\hat{t}\) defined by

\[
\hat{b} = \frac{k^i \times \hat{n}}{|k^i \times \hat{n}|}; \quad \hat{t} = \hat{n} \times \hat{b}. \tag{113}\]

- In this co-ordinate system \(\hat{b}\) is normal to the plane of incidence, spanned by \(k^i\) and \(\hat{n}\).

- Thus, \(k^i\) has no component along \(\hat{b}\), and one can readily show that neither \(k^r\) nor \(k^t\) has a component along \(\hat{b}\), implying that both \(k^r\) and \(k^t\) lie in the plane of incidence.

- Further:

\[
k^i_k = k^r_k = k^t_k = k_t \tag{114}\]

implying that \(k^i\), \(k^r\), and \(k^t\) have equal components parallel to the interface.

- Using

\[
\hat{n} \times k^q = \hat{n} \times (k_t \hat{t} + k^q_n \hat{n}) = -\hat{b} k_t; \quad q = i, r, t \tag{115}\]

we find

\[
\hat{n} \times k^i = \hat{n} \times k^r \tag{116}\]

\[
\hat{n} \times k^i = \hat{n} \times k^t \tag{117}\]

- Further, using (116), \(|a \times b| = |a||b|\sin\theta\), where \(\theta\) is the angle between the vectors \(a\) and \(b\), and Fig. 9.1, we have

\[
k^i \sin \theta^i = k^r \sin \theta^r. \tag{118}\]

- Since \(k^i = k^r = n_1 k_0\), the reflection law becomes

\[
\theta^i = \theta^r \tag{119}\]

which in (116) is given in vectorial form.

- From (117) and Fig. 9.1 we obtain the refraction law or Snell’s law

\[
k^i \sin \theta^i = k^t \sin \theta^t \tag{120}\]

which by using \(k^i = n_1 k_0\) and \(k^t = n_2 k_0\), becomes

\[
n_1 \sin \theta^i = n_2 \sin \theta^t. \tag{121}\]
10.2 Generalisation of the reflection law and Snell’s law

The reflection law and Snell’s law (the refraction law) can be generalised to include non-planar waves that are incident upon a non-planar interface.

This is illustrated in Fig. 9, where the field from a point source propagates towards a curved interface.

Suppose now that the distance from the point source to the interface is much larger than the wavelength. Then at each point on the interface we may consider the incident wave to be a plane wave locally, and we may replace the interface locally by the tangent plane through the point in question.

Then we can use Snell’s law and the reflection law as derived for a plane wave that is incident upon a plane interface, as illustrated in Fig. 9.

10.3 Reflection and refraction of plane electromagnetic waves

In the derivation of the reflection law and Snell’s law we only used that $k^q \cdot r - \omega t$ ($q = i, r, t$) shall be the same for $q = i, q = r, and q = t$. Therefore these laws apply to all types of plane waves, i.e. to acoustic, electromagnetic, and elastic waves, etc..

Now we want to determine how much of the energy in an incident electromagnetic plane wave that is reflected and transmitted, and we choose (Fig. 9.1) the $z$ axis in the direction of the interface normal.

If $E$ is normal to the plane of incidence, we have s polarisation (from german, “Senkrecht”) or $TE$ polarisation (“transverse electric” relative to the plane of incidence).

If $E$ is parallel with the plane of incidence, we have $p$ polarisation or $TM$ polarisation, since in this case $B$ is normal to the plane of incidence.

A general plane electromagnetic wave consists of both a $TE$ and a $TM$ component. Thus, each of the incident, reflected, and transmitted fields can be written in the form ($q = i, r, t$)

$$E^q = E^{TEq} + E^{TMq} \quad B^q = B^{TEq} + B^{TMq}$$

$$E^{TEq} = E^{TEq} \frac{k_t \times \hat{e}_z e^{ik^q r}}{k_t}$$

$$E^{TMq} = E^{TMq} \frac{k^q \times (k_t \times \hat{e}_z) e^{ik^q r}}{k^q k_t}$$

$$B^{TEq} = \frac{1}{k_0} E^{TEq} \frac{k^q \times (k_t \times \hat{e}_z) e^{ik^q r}}{k^q k_t}$$

$$B^{TMq} = -\frac{k^q}{k_0} E^{TMq} k^q \times \hat{e}_z e^{ik^q r}$$

where

$$k^i = k_t + k_{z1} \hat{e}_z \quad k_t = k_x \hat{e}_x + k_y \hat{e}_y$$

$$k^r = k_t - k_{z1} \hat{e}_z \quad k^t = k_t + k_{z2} \hat{e}_z$$

$$k^q = \begin{cases} k_1 = n_1 k_0 & \text{for } q = i, r \\ k_2 = n_2 k_0 & \text{for } q = t. \end{cases}$$
The continuity conditions that must be satisfied at the interface \( z = 0 \) are that the tangential components of \( \mathbf{E} \) and \( \mathbf{H} = \frac{1}{\mu} \mathbf{B} \) be continuous, i.e.

\[
\hat{e}_z \times \left\{ \mathbf{E}^{TEi} + \mathbf{E}^{TEr} - \mathbf{E}^{TEt} + \mathbf{E}^{TMi} + \mathbf{E}^{TMr} - \mathbf{E}^{TMt} \right\} = 0 \quad (130)
\]

\[
\hat{e}_z \times \left\{ \frac{1}{\mu_1} \left( \mathbf{B}^{TEi} + \mathbf{B}^{TEr} \right) - \frac{1}{\mu_2} \mathbf{B}^{TEi} + \frac{1}{\mu_1} \left( \mathbf{B}^{TMi} + \mathbf{B}^{TMr} \right) - \frac{1}{\mu_2} \mathbf{B}^{TMi} \right\} = 0. \quad (131)
\]

Further:

\[
\hat{e}_z \times [k^q \times (k^q \times \hat{e}_z)] = (k^q \cdot \hat{e}_z) \hat{e}_z \times k_t \quad (132)
\]

\[
\hat{e}_z \times (k^q \times \hat{e}_z) = k_t. \quad (133)
\]

Substituting from (123)-(126) into the boundary conditions (130)-(131) and using (112) and (122)-(133), we get

\[
k_t \left\{ \mathbf{E}^{TEi} + \mathbf{E}^{TEr} - \mathbf{E}^{TEt} \right\} + \hat{e}_z \times k_t \left\{ \frac{k_{z1}}{k_1} \mathbf{E}^{TMi} - \frac{k_{z1}}{k_1} \mathbf{E}^{TMr} = \frac{k_{z2}}{k_2} \mathbf{E}^{TMt} \right\} = 0 \quad (134)
\]

\[
\hat{e}_z \times k_t \left\{ \frac{1}{\mu_1} \left( \frac{k_{z1}}{k_0} \mathbf{E}^{TEi} - \frac{k_{z1}}{k_0} \mathbf{E}^{TEr} \right) - \frac{1}{\mu_2} \frac{k_{z2}}{k_0} \mathbf{E}^{TEi} \right\}
+k_t \left\{ \frac{1}{\mu_1} \left( -\frac{k_{z1}}{k_0} \mathbf{E}^{TMi} - \frac{k_{z1}}{k_0} \mathbf{E}^{TMr} \right) - \frac{k_{z2}}{k_0} \mathbf{E}^{TMi} \right\} = 0. \quad (135)
\]

Since \( k_t \) and \( \hat{e}_z \times k_t \) are orthogonal vectors, the expression inside each of the \{\} parentheses in (134) and (135) vanish, i.e.

\[
1 + R^{TE} = T^{TE} \quad ; \quad 1 - R^{TE} = \frac{k_{z2} \mu_1}{k_1 \mu_2} T^{TE} \quad (136)
\]

\[
1 - R^{TM} = \frac{k_{z2} k_1}{k_1 k_2} T^{TM} \quad ; \quad 1 + R^{TM} = \frac{k_{z2} \mu_1}{k_1 \mu_2} T^{TM} \quad (137)
\]

where the reflection and transmission coefficients are given by

\[
R^p = \frac{E^{pr}}{E^{pi}} \quad ; \quad T^p = \frac{E^{pt}}{E^{pi}} \quad (p = TE, TM). \quad (138)
\]

The two equations in (136) have the following solution

\[
R^{TE} = \frac{\mu_2 k_{z1} - \mu_1 k_{z2}}{\mu_2 k_{z1} + \mu_1 k_{z2}} \quad ; \quad T^{TE} = \frac{2 \mu_2 k_{z1}}{\mu_2 k_{z1} + \mu_1 k_{z2}} \quad (139)
\]

whereas the two equations in (137) give

\[
R^{TM} = \frac{k_{z2}^2 \mu_1 k_{z1} - k_{z1}^2 \mu_2 k_{z2}}{k_{z2}^2 \mu_1 k_{z1} + k_{z1}^2 \mu_2 k_{z2}} \quad ; \quad T^{TM} = \frac{2 k_1 k_2 \mu_1 k_{z1}}{k_{z2}^2 \mu_1 k_{z1} + k_{z1}^2 \mu_2 k_{z2}}. \quad (140)
\]

Note that upon reflection and refraction there is no coupling between \( TE \) and \( TM \) waves.

From Fig. 9.1 it follows that

\[
k_{z1} = k_1 \cdot \hat{e}_z = k_1 \cos \theta^i \quad ; \quad k_{z2} = k_1 \cdot \hat{e}_z = k_2 \cos \theta^t \quad (141)
\]

so that if \( \mu_1 = \mu_2 = 1 \) we have

\[
T^{TM} = \frac{2 n_1 \cos \theta^i}{n_2 \cos \theta^i + n_1 \cos \theta^t} \quad ; \quad R^{TM} = \frac{n_2 \cos \theta^i - n_1 \cos \theta^t}{n_2 \cos \theta^i + n_1 \cos \theta^t} \quad (142)
\]

\[
T^{TE} = \frac{2 n_1 \cos \theta^i}{n_1 \cos \theta^i + n_2 \cos \theta^t} \quad ; \quad R^{TE} = \frac{n_1 \cos \theta^i - n_2 \cos \theta^t}{n_1 \cos \theta^i + n_2 \cos \theta^t}. \quad (143)
\]

These expressions are called the Fresnel formulas. Using Snell’s law (121), we can rewrite them as

\[
T^{TM} = \frac{2 \sin \theta^i \cos \theta^t}{\sin (\theta^i + \theta^t) \cos (\theta^i - \theta^t)} \quad ; \quad R^{TM} = \frac{\tan(\theta^i - \theta^t)}{\tan(\theta^i + \theta^t)} \quad (144)
\]

\[
T^{TE} = \frac{2 \sin \theta^t \cos \theta^i}{\sin (\theta^i + \theta^t) \cos (\theta^i - \theta^t)} \quad ; \quad R^{TE} = \frac{-\sin(\theta^i - \theta^t)}{\sin(\theta^i + \theta^t)}. \quad (145)
\]

At normal incidence where \( \theta^i = \theta^t = 0 \), we get from (142) and (143)

\[
T^{TE} = T^{TM} = \frac{2}{n + 1} \quad ; \quad R^{TM} = -R^{TE} = \frac{n - 1}{n} \quad ; \quad n = \frac{n_2}{n_1}. \quad (146)
\]

The fact that \( R^{TM} = -R^{TE} \) at normal incidence follows from the way in which \( \mathbf{E}^{TE} \) and \( \mathbf{E}^{TE} \) are defined. From Fig. 9.4 we see that these two vectors point in opposite directions at normal incidence.
10.4 Reflectance and transmittance

- Let the angle between $\mathbf{E}^q$ and the plane of incidence spanned by $\mathbf{k}^q$ and $\hat{\mathbf{e}}^{TMq}$, be $\alpha^q$ [see Fig. 11], so that

$$\mathbf{E}^q = \hat{\mathbf{e}}^{TE} E^q \sin \alpha^q + \hat{\mathbf{e}}^{TEm} E^q \cos \alpha^q. \tag{149}$$

- Further, let $J^i$, $J^r$, and $J^t$ denote the energy flows of respectively the incident, reflected, and transmitted fields per unit area of the interface, so that

$$J^{pq} = S^{pq} \cos \theta_q; \quad p = TE, TM \quad (q = i, r, t) \tag{150}$$

where $S^{pq}$ is the absolute value of the Poynting vector, given by

$$S^{pq} = \frac{c}{4\pi} |\mathbf{E}^{pq} \times \mathbf{H}^{pq}| = \frac{c}{4\pi} E^{pq} H^{pq} = \frac{c}{4\pi} \sqrt{\varepsilon^q (E^{pq})^2}. \tag{151}$$

- Here we have used the relation $\sqrt{\varepsilon^q E^{pq}} = \sqrt{n^q} H^{pq}$.

- The reflectance $R^p (p = TE, TM)$ is the ratio between the reflected and the incident energy flows, i.e. [cf. (150)-(151)]

$$R^{TM} = \frac{J^{TMr}}{J^{TEi}} = \frac{|E^{TMr}|^2}{|E^{TEi}|^2} = (R^T)^2, \tag{152}$$

$$R^{TE} = \frac{J^{TEr}}{J^{TEi}} = \frac{|E^{TEr}|^2}{|E^{TEi}|^2} = (R^E)^2, \tag{153}$$

- Thus, the reflectance $R^p$ is equal to the square of reflection coefficient $R^p$.

- The transmittance $T^p (p = TE, TM)$ is the ratio between the transmitted and the incident energy flows, i.e. [cf. (150)-(151)]

$$T^{TM} = \frac{J^{TMt}}{J^{TMi}} = \frac{n_2 \mu_1 \cos \theta^l}{n_1 \mu_2 \cos \theta^l} (TM)^2 \tag{154}$$

$$T^{TE} = \frac{J^{TEt}}{J^{TEi}} = \frac{n_2 \mu_1 \cos \theta^l}{n_1 \mu_2 \cos \theta^l} (TE)^2. \tag{155}$$

- Thus, the transmittance $T^p$ is proportional to the square of the transmission coefficient $T^p$ (p = TE, TM).
When $\mu_2 = \mu_1 = 1$, we find on substitution from (144)-(145) into (152)-(155):

$$R^{TM} = \frac{\tan^2(\theta^i - \theta^t)}{\tan^2(\theta^i + \theta^t)}$$  \hspace{1cm} (156)$$

$$R^{TE} = \frac{\sin^2(\theta^i - \theta^t)}{\sin^2(\theta^i + \theta^t)}$$  \hspace{1cm} (157)$$

$$T^{TM} = \frac{\sin 2\theta^i \sin 2\theta^t}{\sin^2(\theta^i + \theta^t) \cos^2(\theta^i - \theta^t)}$$  \hspace{1cm} (158)$$

$$T^{TE} = \frac{\sin 2\theta^i \sin 2\theta^t}{\sin^2(\theta^i + \theta^t)}.$$  \hspace{1cm} (159)$$

Using these formulas, one can show that

$$R^{TM} + T^{TM} = 1; \quad R^{TE} + T^{TE} = 1.$$  \hspace{1cm} (160)$$

Thus, for each of the two polarisations the sum of the reflected energy and the transmitted energy is equal to the incident energy.

From (150) og (151) we have

$$J^{pi} = \frac{c}{4\pi} \sqrt{\frac{\varepsilon_1}{\mu_1}} |E^{pi}|^2 \cos \theta^i$$  \hspace{1cm} (161)$$

which by the use of (149) gives

$$J^{TEi} = \cos \theta^i \frac{c}{4\pi} \sqrt{\frac{\varepsilon_1}{\mu_1}} |E^{TEi}|^2 = \cos \theta^i \frac{c}{4\pi} \sqrt{\frac{\varepsilon_1}{\mu_1}} E^2 \sin^2 \alpha^i$$  \hspace{1cm} (162)$$

$$J^{TMi} = \cos \theta^i \frac{c}{4\pi} \sqrt{\frac{\varepsilon_1}{\mu_1}} |E^{TMi}|^2 = \cos \theta^i \frac{c}{4\pi} \sqrt{\frac{\varepsilon_1}{\mu_1}} E^2 \cos^2 \alpha^i.$$  \hspace{1cm} (163)$$

But since the total energy flow is given by

$$J^i = \cos \theta^i \frac{c}{4\pi} \sqrt{\frac{\varepsilon_1}{\mu_1}} E^2$$  \hspace{1cm} (164)$$

we find

$$J^{TEi} = J^i \sin^2 \alpha^i; \quad J^{TMi} = J^i \cos^2 \alpha^i.$$  \hspace{1cm} (165)$$

Thus:

$$\mathcal{R} = \frac{J^r}{J^i} = \frac{J^{TMr} + J^{TER}}{J^i} = \frac{J^{TMr}}{J^{TMi}} \cos^2 \alpha^i + \frac{J^{TER}}{J^{TER}} \sin^2 \alpha^i$$  \hspace{1cm} (166)$$

which gives

$$\mathcal{R} = R^{TM} \cos^2 \alpha^i + R^{TE} \sin^2 \alpha^i.$$  \hspace{1cm} (167)$$

Similarly we find

$$T = T^{TM} \cos^2 \alpha^i + T^{TE} \sin^2 \alpha^i.$$  \hspace{1cm} (168)$$

At normal incidence, $\theta^i = \theta^t = 0$, and the distinction between $TE$ and $TM$ polarisation disappears. From (152)-(155) combined with (142)-(143), we find (when $\mu_1 = \mu_2 = 1$)

$$\mathcal{R} = R^{TM} = R^{TE} = (R^{TE})^2 = (R^{TM})^2 = \left(\frac{n-1}{n+1}\right)^2; \quad n = \frac{n_2}{n_1},$$  \hspace{1cm} (169)$$

$$T = T^{TM} = T^{TE} = (T^{TE})^2 = (T^{TM})^2 = \frac{4n}{(n+1)^2}; \quad n = \frac{n_2}{n_1}.$$  \hspace{1cm} (170)$$

When $n \to 1$, we see that $\mathcal{R} \to 0$ and $T \to 1$, as expected.

Similarly, from (156)-(159): $R^{TM} \to 0$, $R^{TE} \to 0$, $T^{TM} \to 1$, and $T^{TE} \to 1$ when $n \to 1$.

### 10.5 Brewster’s law

- From (156) it follows that $R^{TM} = 0$ når $\theta^iB + \theta^tB = \pi/2$, since then $\tan(\theta^iB + \theta^tB) = \infty$.
- From Snell’s law (121):

$$n_2 \sin \theta^iB = n_2 \sin \left(\frac{\pi}{2} - \theta^iB\right) = n_2 \cos \theta^iB = n_1 \sin \theta^iB.$$  \hspace{1cm} (171)$$

Thus, $R^{TM} = 0$ when $\theta^iB$ is given by

$$\tan \theta^iB = \frac{n_2}{n_1} = n.$$  \hspace{1cm} (172)$$
Figure 12: Illustration of Brewster’s law.

• $\theta^B$ is called the polarisation angle or the Brewster angle.

• When the angle of incidence is equal to $\theta^B$, the E vector of the reflected light has no component in the plane of incidence (Fig. 9.6). This fact is exploited in sunglasses with polarisation filter.

• The filter is oriented such that only light that is polarised vertically (Fig. 9.6) is transmitted. Thus, one avoids to a certain degree annoying reflections from e.g. a water surface.

• Note that $k_r \cdot k_t = 0$, i.e. $k_r$ and $k_t$ are normal to one another when $\theta_i = \theta^B$, as shown in Fig. 12.

10.6 Unpolarised light (natural light)

• For natural light, e.g. light from an incandescent lamp, the direction of the E vector varies very rapidly in arbitrary or irregular manner, so that no particular direction is given preference.

• The average reflectance $\bar{R}$ is obtained by averaging over all directions $\alpha$. Since the average value of both $\sin^2 \alpha$ and $\cos^2 \alpha$ is $\frac{1}{2}$, (165) gives

$$\bar{J}^{TM_i} = J^T \cos^2 \alpha = J^E \sin^2 \alpha = \frac{1}{2} J^i.$$  

• For the reflected components we find

$$\bar{J}^{TM_r} = \frac{J^{TM_r}}{\bar{J}^{TM_i}} \cdot \bar{J}^{TM_i} = \frac{J^{TM_r}}{2 J^{TM_i}} \cdot \frac{1}{2} J^i = \frac{1}{2} R^{TM} J^i,$$

$$\bar{J}^{TE_r} = \frac{J^{TE_r}}{\bar{J}^{TE_i}} \cdot \bar{J}^{TE_i} = \frac{1}{2} J^i = \frac{1}{2} R^{TE} J^i$$  (175)

which shows that the degree of polarisation for the reflected light can be defined as

$$P^r = \frac{|R^{TM} - R^{TE}|}{R^{TM} + R^{TE}} = \frac{|J^{TM_r} - J^{TE_r}|}{J^{TM_r} + J^{TE_r}}.$$  (176)

• The average reflectance is given by

$$\bar{R} = \frac{\bar{J}}{\bar{J}'} = \frac{\bar{J}^{TM_r} + \bar{J}^{TE_r}}{\bar{J}'} = \frac{\bar{J}^{TM_r}}{2 J^{TM_i}} + \frac{\bar{J}^{TE_r}}{2 J^{TE_i}} = \frac{1}{2} \left( R^{TM} + R^{TE} \right)$$  (177)

so that the degree of polarisation becomes

$$P^r = \frac{1}{\bar{R}} \frac{1}{2} |R^{TM} - R^{TE}|$$  (178)

where $|R^{TM} - R^{TE}|$ is called the polarised part of the reflected light.

• Similarly, for the transmitted light:

$$\bar{T} = \frac{1}{2} (\bar{T}^{TM} + \bar{T}^{TE}) ; \quad P^t = \frac{1}{\bar{T}} \frac{1}{2} |T^{TM} - T^{TE}|.$$  (179)

10.7 Rotation of the plane of polarisation upon reflection and refraction

• Note that if the incident light is linearly polarised, then also the reflected and the transmitted light will be linearly polarised, since the phases only change by 0 or $\pi$.

• But the planes of polarisation for the reflected and the transmitted light are rotated in opposite directions relative to the polarisation plane of the incident light.

• The angles $\alpha^i$, $\alpha^r$, and $\alpha^t$ that the planes of polarisation of the incident, reflected, and transmitted light form with the plane of incidence, are given by [cf. Fig. 9.5]

$$\tan \alpha^i = \frac{E^{TE_i}}{E^{TM_i}}$$  (180)

$$\tan \alpha^r = \frac{E^{TE_r}}{E^{TM_r}} = \frac{E^{TE_r}}{E^{TM_r}} \frac{E^{TE_i}}{E^{TM_i}} = \frac{R^{TE}}{R^{TM}} \tan \alpha^i$$  (181)
These two inequalities imply that upon reflection the plane of polarisation is rotated away from the plane of incidence, whereas upon transmission it is rotated towards the plane of incidence.

When \( \theta^i = \theta^B \), so that \( \theta^i + \theta^B = \pi \), then \( \tan \alpha^r = \infty \). Thus, we have \( \alpha^r = \pi/2 \) in accordance with Brewster’s law.

### 10.8 Total reflection

- Snell’s law (121) can be written in the form
  \[
  \sin \theta^t = \frac{\sin \theta^i}{n} \quad ; \quad n = \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_2 \mu_2}{\varepsilon_1 \mu_1}}.
  \]  
  (187)

- Thus, if \( n < 1 \), we get \( \sin \theta^t = 1 \) when \( \theta^i = \theta^ic \), where
  \[
  \sin \theta^ic = n.
  \]  
  (188)

- When \( \theta^i = \theta^ic \), we get \( \theta^t = \pi/2 \), so that the transmitted light propagates along the interface.

- When \( \theta^i \geq \theta^ic \), we have total reflection.

When \( \theta^i > \theta^ic \), \( \sin \theta^t > 1 \), and \( \theta^t \) is complex:
  
  \[
  \cos \theta^t = \sqrt{1 - \sin^2 \theta^t} = i \sqrt{\frac{\sin^2 \theta^t}{n^2} - 1} = i \sqrt{\frac{\sin^2 \theta^i - n^2}{n}}
  \]  
  (189)

- The electric field in medium 2 is
  
  \[
  E_{p}^t = T^p E_{p}^i e^{ik^t \cdot r - \omega t} \quad (p = TE, TM)
  \]  
  (190)

where

\[
 e^{ik^t \cdot r} = e^{i(k_x x + k_y y)} e^{-|k_z z|} \quad ; \quad |k_z z| = \frac{k_2}{n} \sqrt{\sin^2 \theta^i - n^2}.
\]

Thus, \( E_{p}^t \) represents a wave that propagates along the interface and is exponentially damped with the distance z into medium 2.

- One can show that the time average of the z component of the Poynting vector is zero. Thus, there is no energy transport through the interface.

- Using Snell’s law the reflection coefficients

  \[
  R_{TM}^c = \frac{\sin \theta^i \cos \theta^i - \sin \theta^t \cos \theta^t}{\sin \theta^i \cos \theta^i + \sin \theta^t \cos \theta^t}
  \]  
  \[
  R_{TE}^c = \frac{-\sin \theta^i \cos \theta^i + \sin \theta^t \cos \theta^t}{\sin \theta^i \cos \theta^i + \sin \theta^t \cos \theta^t}
  \]  
  (192)
become
\[ R^{TM} = \frac{n^2 \cos \theta^i - i \sqrt{\sin^2 \theta^i - n^2}}{n^2 \cos \theta^i + i \sqrt{\sin^2 \theta^i - n^2}} \]
(194)
\[ R^{TE} = \frac{\cos \theta^i - i \sqrt{\sin^2 \theta^i - n^2}}{\cos \theta^i + i \sqrt{\sin^2 \theta^i - n^2}}. \]
(195)

• Since both reflection coefficients are of the form \(z/z^*\), where \(z\) is a complex number, we have
\[ |R^{TM}| = |R^{TE}| = 1. \]
(196)

• Thus, for each polarisation the intensity of the totally reflected light is equal to the intensity of the incident light.

• But the phase is altered upon total reflection.

• Letting
\[ R^p = \frac{E^p_r}{E^p_i} = e^{i \delta^p} = \frac{z^p}{z^{p*}} = e^{2i \alpha^p} \quad (p = TE, TM) \]
(197)
where
\[ z^{TM} = n^2 \cos \theta^i - i \sqrt{\sin^2 \theta^i - n^2} = |z^{TM}| e^{i \alpha^{TM}} \]
(198)
\[ z^{TE} = \cos \theta^i - i \sqrt{\sin^2 \theta^i - n^2} = |z^{TE}| e^{i \alpha^{TE}} \]
(199)
we have
\[ \tan \alpha^{TM} = \tan \left(\frac{1}{2} \delta^{TM}\right) = -\frac{\sqrt{\sin^2 \theta^i - n^2}}{n^2 \cos \theta^i} \]
(200)
\[ \tan \alpha^{TE} = \tan \left(\frac{1}{2} \delta^{TE}\right) = -\frac{\sqrt{\sin^2 \theta^i - n^2}}{\cos \theta^i}. \]
(201)

• The relative phase difference \(\delta = \delta^{TE} - \delta^{TM}\) becomes
\[ \tan \left(\frac{1}{2} \delta\right) = \tan \left(\frac{1}{2} \delta^{TE}\right) - \tan \left(\frac{1}{2} \delta^{TM}\right) = \tan \left(\frac{1}{2} \delta\right) = \frac{\cos \theta^i \sqrt{\sin^2 \theta^i - n^2}}{\sin^2 \theta^i}. \]
(202)

• Since \(\delta = 0\) for \(\theta^i = \frac{\pi}{2}\) (grazing incidence) and \(\theta^i = \theta^i_{ic}\) (critical angle of incidence), between these two values there is an angle of incidence \(\theta^i = \theta^i_{im}\) which gives a maximum phase difference \(\delta = \delta^m\).

• \(\theta^i_{im}\) is determined by \(\frac{d\delta}{d\theta^i} \bigg|_{\theta^i_{im}} = 0\), and we find
\[ \sin^2 \theta^i_{im} = \frac{2 n^2}{1 + n^2} \quad ; \quad \tan \left(\frac{1}{2} \delta^m\right) = \frac{1 - n^2}{2n}. \]
(203)

• If the phase difference \(\delta\) is equal to \(\pm \frac{\pi}{2}\) and in addition \(E^{TMi} = E^{TEi}\), the totally reflected light will be circularly polarised.

• By choosing the angle \(\alpha^i\) between the polarisation plane and the plane of incidence equal to \(45^\circ\), we make \(E^{TMi}\) equal to \(E^{TEi}\).

• To obtain \(\delta = \frac{\pi}{2}\), \(\frac{\delta^m}{\delta^T} \geq \frac{\pi}{4}\), implying that \(\frac{n_1}{n_2}\) must exceed 2.41.
11 Boundary Value Problems

11.1 Angular-spectrum representations

- Consider a 3D scalar wave field \( \hat{u}(\mathbf{r}, t) \) in a linear, homogeneous, non-dispersive, and isotropic medium.

- In source free regions of space, \( \hat{u}(\mathbf{r}, t) \) is a solution of the scalar wave equation

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \hat{u}(\mathbf{r}, t) = 0
\]

(204)

where \( c \) is the phase velocity in the medium.

- To study monochromatic or quasi-monochromatic phenomena we Fourier decompose \( \hat{u}(\mathbf{r}, t) \):

\[
\hat{u}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(\mathbf{r}, \omega)e^{-i\omega t} \, d\omega.
\]

(205)

- Substitution of (205) in (204) shows that \( u(\mathbf{r}, \omega) \) satisfies the Helmholtz equation:

\[
\left( \nabla^2 + k^2 \right) u(\mathbf{r}, \omega) = 0 \quad ; \quad k = \frac{\omega}{c}
\]

(206)

- Let the source lie in the half-space \( z < 0 \), and let the field be known in the plane \( z = 0 \).

- Hereafter we consider only one single Fourier component of the field and write for simplicity \( u(\mathbf{r}) \) instead of \( u(\mathbf{r}, \omega) \).

- Through Fourier decomposition of \( u(\mathbf{r}) \) with respect to \( x \) and \( y \) we have

\[
u(\mathbf{r}) = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{u}(k_x, k_y; z)e^{i(k_x x + k_y y)} \, dk_x dk_y.
\]

(207)

- It can readily be shown that

\[
u(k_x, k_y; z) = U(k_x, k_y)e^{ik_z z}
\]

(208)

so that he field in the half-space \( z > 0 \) is given by

\[
u(\mathbf{r}) = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(k_x, k_y)e^{ik \cdot r} \, dk_x dk_y,
\]

(209)

\[
k \cdot r = k_x x + k_y y + k_z z
\]

(210)

\[
k_z = \begin{cases} \sqrt{k_x^2 - k_y^2 - k_z^2} & \text{for } k^2 \geq k_x^2 + k_y^2 \\ i\sqrt{k_x^2 + k_y^2 - k_z^2} & \text{for } k^2 < k_x^2 + k_y^2 \end{cases}
\]

(211)

- Equation (209) is called an angular-spectrum representation, since the field \( u(\mathbf{r}) \) is given as a sum of plane waves \( \exp(ik \cdot \mathbf{r}) \) that propagate in various directions \( \hat{s} \), where \( \hat{s} = k/k \).

- When \( k^2 \geq k_x^2 + k_y^2 \), then \( \exp(ik \cdot \mathbf{r}) \) is a homogeneous plane wave.

- When \( k^2 < k_x^2 + k_y^2 \), then \( \exp(ik \cdot \mathbf{r}) \) is an inhomogeneous or evanescent plane wave.

- An evanescent plane wave propagates in a direction normal to the \( z \) axis and decays exponentially with increasing \( z \).

- The amplitude \( U(k_x, k_y) \) of each individual plane wave in (209), called the angular spectrum, can be determined by setting \( z = 0 \) in (209):

\[
U(k_x, k_y) = \mathcal{F}\{u(x, y, 0)\} = \int_{-\infty}^{\infty} u(x, y, 0)e^{-i(k_x x + k_y y)} \, dx dy.
\]

(212)

- Thus, the angular spectrum \( U(k_x, k_y) \) is the Fourier transform of the field in the plane \( z = 0 \).

Figure 14: A source in the half-space \( z < 0 \) radiates a field that is assumed to be known in the plane \( z = 0 \). The field in the half-space \( z > 0 \) is to be determined.
If we know the normal derivative \( \frac{\partial u}{\partial z} \) of the field in the plane \( z = 0 \) instead of \( u \), then the field in the half-space \( z > 0 \) is given by

\[
u(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} U(k_x, k_y) \frac{e^{ik\cdot r}}{ik_z} dk_x dk_y,
\]

(213)

where \( U(k_x, k_y) \) is the Fourier transform of \( \frac{\partial u}{\partial z} \) in the plane \( z = 0 \).

To distinguish between these two solutions, we denote them by \( u_I \) and \( u_{II} \):

\[
u_I(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} U(k_x, k_y) e^{ik\cdot r} dk_x dk_y,
\]

(214)

\[
u = \int_{-\infty}^{\infty} u(x, y, 0) e^{-i(k_x x + k_y y)} dx dy,
\]

(215)

\[
u_{II}(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} U'(k_x, k_y) \frac{e^{ik\cdot r}}{ik_z} dk_x dk_y,
\]

(216)

\[
U'(k_x, k_y) = \int_{-\infty}^{\infty} \frac{\partial u(\mathbf{r})}{\partial z} \bigg|_{z=0} e^{-i(k_x x + k_y y)} dx dy.
\]

(217)

Note that we have made no approximations, so that both solutions are exact. In other words, \( u_I \) and \( u_{II} \) are identical.

11.2 Rayleigh-Sommerfeld’s and Kirchhoff’s diffraction integrals

- Using the convolution theorem for two-dimensional Fourier transform pairs, we find

\[
u_I(\mathbf{r}) = \int_{-\infty}^{\infty} u(x', y', 0) h(x - x', y - y') dx' dy'
\]

(218)

where

\[
h(x, y, z) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} e^{ik\cdot r} dk_x dk_y = -\frac{1}{2\pi} \frac{\partial}{\partial z} \left\{ \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{ik\cdot r} dk_x dk_y \right\}.
\]

(219)

- Weyl’s plane-wave expansion of a spherical wave is given by

\[
\frac{e^{ikr}}{r} = \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{ik\cdot r} dk_x dk_y.
\]

(220)

- Thus, (218) gives Rayleigh-Sommerfeld’s first diffraction integral:

\[
u_I(\mathbf{r}) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} u(x', y', 0) \frac{\partial}{\partial z} \left( \frac{e^{ikR}}{R} \right) dx' dy'
\]

(221)

where

\[
R = [(x - x')^2 + (y - y')^2 + z_2]^{1/2}.
\]

(222)

- Similarly, Rayleigh-Sommerfeld’s second diffraction integral is given by

\[
u_{II}(\mathbf{r}) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{\partial u(x', y', z)}{\partial z} \right]_{z=0} \frac{e^{ikR}}{R} dx' dy'.
\]

(223)

- Since we have made no approximations, both (221) and (223) are exact solutions.
Kirchhoff’s diffraction integral is half the sum of the two Rayleigh-Sommerfeld integrals, i.e.

$$u_K(r) = \frac{1}{2}[u_I(r) + u_{II}(r)]. \quad (224)$$

### 12 Diffraction problems

#### 12.1 Fresnel and Fraunhofer diffraction

A source in the half-space $z < 0$ radiates a field $u^i$ that is diffracted through an aperture $A$ in the plane $z = 0$. The field in the observation point $(x_2, y_2, z_2)$ in the half-space $z > 0$ is to be determined.

- Consider a field $u^i$ that is generated by sources in the half-space $z < 0$, and that propagates towards an aperture in the plane $z = 0$ (see Fig. 11.1).

- To determine the diffracted field in the half-space $z > 0$ we use Rayleigh-Sommerfeld’s first diffraction integral and a variant of the Kirchhoff approximation.

- Our variant of the Kirchhoff approximation implies that we replace the actual field in the plane $z = 0^+$ by the incident field inside the aperture and by zero outside the aperture.

- Thus, we have

$$u_I(x_2, y_2, z_2 > 0) = -\frac{1}{2\pi} \int_A u^i(x, y, 0) \frac{\partial}{\partial z_2} \left( \frac{e^{ikR_2}}{R_2} \right) dxdy$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{\infty} t(x, y)u^i(x, y, 0) \frac{\partial}{\partial z_2} \left( \frac{e^{ikR_2}}{R_2} \right) dxdy \quad (225)$$

where $A$ is the aperture area, and $t(x, y)$ has the value 1 inside the aperture and the value 0 outside the aperture.

- $R_2$ is the distance from an integration point $(x, y, 0)$ in the aperture plane to the observation point $(x_2, y_2, z_2)$:

$$R_2 = \sqrt{(x-x_2)^2 + (y-y_2)^2 + z_2^2}. \quad (226)$$

- Carrying out the differentiation with respect to $z_2$ in (225), we find

$$\frac{\partial}{\partial z_2} \left( \frac{e^{ikR_2}}{R_2} \right) = -\frac{z_2}{R_2^2} \frac{e^{ikR_2}}{R_2} \left( 1 + \frac{i}{kR_2} \right). \quad (227)$$

- Assuming that $kR_2 >> 1$, and introducing the paraxial approximation

$$\frac{z_2}{R_2} \simeq 1 \quad (228)$$

we have

$$\frac{\partial}{\partial z_2} \left( \frac{e^{ikR_2}}{R_2} \right) \simeq \frac{ik}{z_2} e^{ikR_2}. \quad (229)$$

- Note that we have used the paraxial approximation only in the amplitude factor on the right-hand side of (227).

#### 12.2 Fresnel diffraction

- Since $\exp(ikR_2)$ is a rapidly oscillating function compared with $R_2$, we introduce the Fresnel approximation:

$$R_2 = z_2 \sqrt{1 + \frac{(x-x_2)^2 + (y-y_2)^2}{z_2^2}} \simeq z_2 \left( 1 + \frac{1}{2} \frac{(x-x_2)^2 + (y-y_2)^2}{z_2^2} \right) \quad (230)$$

which requires that

$$\frac{(x-x_2)^2 + (y-y_2)^2}{z_2^2} \leq 1. \quad (231)$$

- Using both the Fresnel and the paraxial approximations, we obtain Fresnel diffraction:
Consider a plane wave

\[ u_I = \frac{C'}{i\lambda z_2} \int_{-\infty}^{\infty} u'(x, y, 0) t(x, y) \exp \left[ ik \left( \frac{x^2 + y^2}{2z_2} - \frac{x^2 + y^2}{z_2} \right) \right] dxdy \]  

(232)

where

\[ C' = e^{ik\Phi}; \quad \Phi = z_2 + \frac{x^2 + y^2}{2z_2}. \]  

(233)

12.3 Fraunhofer diffraction

- Let the observation distance \( z_2 \) be so large that we may set the factor \( \exp[ik(x^2 + y^2)/2z_2] \) in (232) equal to 1.
- Then we have Fraunhofer diffraction, which requires that

\[ \frac{k(x^2 + y^2)}{2z_2} \ll 1. \]  

(234)

- From (232) it follows that the diffracted field becomes equal to the Fourier transform of the field in the aperture:

\[ u_I = \frac{C'}{i\lambda z_2} A(k_x, k_y); \quad k_x = \frac{kx}{z_2}; \quad k_y = \frac{ky}{z_2} \]  

(235)

where

\[ a(x, y) = u(x, y, 0) t(x, y) \]  

(236)

\[ A(k_x, k_y) = \mathcal{F}\{a(x, y)\} = \int_{-\infty}^{\infty} a(x, y) e^{-i(k_xx + k_yy)} dxdy. \]  

(237)

12.4 Circular aperture

- Consider a plane wave \( u' \) that is normally incident upon a circular aperture of radius \( a \) in the plane \( z = 0 \).
- Thus, \( u' = e^{ikz} \), \( u'(x, y, 0) = 1 \), and (see Fig. 11.2)

\[ t(x, y) = \begin{cases} 
1 & \text{for } x^2 + y^2 \leq a^2 \\
0 & \text{for } x^2 + y^2 > a^2.
\end{cases} \]  

(238)

Fresnel diffraction. Introducing dimensionless co-ordinates \( u \) and \( v \) defined by

\[ v = k \left( \frac{a}{z_2} \right) r = \frac{2\pi}{\lambda} \left( \frac{a}{z_2} \right) r; \quad u = k \frac{a^2}{z_2} = \frac{2\pi}{\lambda} \left( \frac{a}{z_2} \right)^2 z_2 \]  

(239)

we obtain in the Fresnel approximation:

\[ u_I = -2iC\frac{\pi a^2}{\lambda z_2} \int_0^1 J_0(\nu t)e^{i\frac{u}{r^2}} dt \]  

(240)

where the zeroth-order Bessel function is given by

\[ J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \cos(\phi - \beta)} d\phi. \]  

(241)

Fraunhofer diffraction. As \( z_2 \to \infty \), \( u \) in (239) approaches zero, so that (240) gives

\[ u_I = 2C \int_0^1 J_0(\nu t) dt = C \frac{2J_1(\nu)}{\nu}; \quad C = \frac{\pi a}{i\lambda z_2} \exp\left\{ ik \left( z_2 + \frac{r^2}{2z_2} \right) \right\}. \]  

(242)

- Then the intensity distribution becomes

\[ I = |u_I|^2 = |C|^2 \left( \frac{2J_1(\nu)}{\nu} \right)^2; \quad |C|^2 = \left( \frac{\pi a^2}{\lambda z_2} \right)^2. \]  

(243)

- This intensity distribution is called the Airy diffraction pattern.
• The first zero of \(J_1(x)\) occurs when \(x = 3.83\), so that the diameter of the Airy pattern is determined by

\[
v = v_0 = \frac{2\pi}{\lambda} \frac{a}{z_2} r_0 = 3.83
\]

or

\[
D = 2r_0 = \frac{3.83 z_2}{\pi} \frac{a}{\lambda} = 1.22 \left(\frac{z_2}{a}\right) \lambda.
\]  (244)

• Introducing the f-number, defined by \(F = \frac{z_2^2}{2a}\), we get

\[
D = 2.44 F \lambda.
\]  (245)

• Note that the diameter \(D\) of the Airy disc is inversely proportional to \(a\).

**Axial field.** At axial observation points \(r = 0\), we have in the Fresnel approximation:

\[
u_I(u,0) = 2 \exp\left\{i \left[k \frac{z_2}{2} + \frac{a}{4} \right] \right\} \sin\left(\frac{u}{4}\right) ; \quad I = 4 \sin^2\left(\frac{u}{4}\right).
\]  (246)

• In the limit of Fraunhofer diffraction (\(u \to 0\)), we have:

\[
u_I(0,0) = \frac{\pi a^2}{\lambda z_2} \exp\left\{i \left[k \frac{z_2}{2} - \frac{\pi}{2}\right] \right\} ; \quad I(0,0) = \left(\frac{\pi a^2}{\lambda z_2}\right).
\]  (247)

in agreement with the results obtained from (242) and (243) in the limit as \(v \to 0\).

• As \(z_2 \to 0\), the axial intensity oscillates very rapidly. To see this, we note that

\[
\|\Delta u\| = 2\pi \left(\frac{a}{z_2}\right)^2 \frac{\|\Delta z_2\|}{\lambda}.
\]  (248)

• Thus, when \(z_2 = a\), a change in \(\|\Delta z_2\|\) of \(4\lambda\) produces a full cycle of \(\sin(u/4)\).

• When \(z_2 = 0.01a\), a change in \(\Delta z_2\) of \(4\lambda \times 10^{-4}\) produces a full cycle of \(\sin(u/4)\).

### 12.5 Rectangular aperture

• Consider a plane wave that is normally incident upon a rectangular aperture in the plane \(z = 0\) with midpoint at \(x = y = 0\) and with sides \(2a\) and \(2b\) in the \(x\) and \(y\) direction, respectively.

• Then (232) gives

\[
u_I = \frac{C'}{i \lambda z_2} \int_{-a}^{a} \exp\left\{i k \left[\frac{x^2}{2z_2} - \frac{x^2}{2z_2}\right] \right\} dx \int_{-b}^{b} \exp\left\{i k \left[\frac{y^2}{2z_2} - \frac{y^2}{2z_2}\right] \right\} dy.
\]  (249)

**Fraunhofer diffraction.** Let us assume that

\[
\frac{ka^2}{2z_2} << 1 \quad \text{og} \quad \frac{kb^2}{2z_2} << 1.
\]  (250)

• Then (249) gives

\[
u_I = \frac{C'}{i \lambda z_2} 4ab \sin(v_a) \sin(v_b) ; \quad v_a = k \frac{a}{z_2} x ; \quad v_b = k \frac{b}{z_2} y.
\]  (251)

• The intensity of the diffraction pattern becomes

\[
I = |\nu_I|^2 = \left(\frac{4ab}{\lambda z_2}\right)^2 \sin^2(v_a) \sin^2(v_b).
\]  (252)

• Since \(\sin(x)\) has its first zeros at \(x = \pm \pi\), the extent of the diffraction pattern between the two first zeros in the \(x\) or \(y\) direction becomes

\[
D_x = \frac{z_2}{a} \lambda ; \quad D_y = \frac{z_2}{b} \lambda.
\]  (253)

• For a square aperture (\(a = b\)) we get

\[
D_x = D_y = \frac{z_2}{a} \lambda
\]  (254)

which is seen to be a little less than the corresponding extent of the Airy disc for a circular aperture of radius \(a\). In the latter case we have according to (244)

\[
D = 1.22 \frac{z_2}{a} \lambda.
\]  (255)
Fresnel diffraction. When the Fraunhofer condition (250) is not satisfied, we have Fresnel diffraction.

- Then we write \( u_I \) in the form
  \[
  u_I = \frac{C'}{i\lambda z_2}I_xI_y
  \]  
  (256)
  where \( I_x \) is given by
  \[
  I_x = a\sqrt{\frac{2}{\pi}} \frac{1}{u_a} \left\{ \int_0^{\alpha_a^+} \cos(\alpha^2)d\alpha - \int_0^{\alpha_a^-} \cos(\alpha^2)d\alpha \right\}
  \]
  with
  \[
  \alpha_a^\pm = -\sqrt{\frac{u_a}{2}} \left( \frac{v_a}{u_a} \mp 1 \right) ; \quad u_a = k \left( \frac{a}{z_2} \right)^2 z_2 ; \quad v_a = k \frac{a}{z_2} x_2.
  \]

- The Fresnel integrals \( C(u) \) and \( S(u) \) are defined as
  \[
  C(u) = \sqrt{\frac{2}{\pi}} \int_0^u \cos(t^2)dt ; \quad S(u) = \sqrt{\frac{2}{\pi}} \int_0^u \sin(t^2)dt.
  \]
  (259)
  Thus, (257) can be expressed in terms of \( C \) and \( S \) in the following manner
  \[
  I_x = a\sqrt{\frac{\pi}{u_a}} e^{-i\frac{\pi}{4}} \left\{ C(\alpha_a^+) - C(\alpha_a^-) + i[S(\alpha_a^+) - S(\alpha_a^-)] \right\}.
  \]
  (260)
  Similarly, we obtain for the integral \( I_y \):
  \[
  I_y = b\sqrt{\frac{\pi}{u_b}} e^{-i\frac{\pi}{4}} \left\{ C(\alpha_b^+) - C(\alpha_b^-) + i[S(\alpha_b^+) - S(\alpha_b^-)] \right\},
  \]
  (261)
  where
  \[
  \alpha_b^\pm = -\sqrt{\frac{u_b}{2}} \left( \frac{v_b}{u_b} \mp 1 \right) ; \quad u_b = k \left( \frac{b}{z_2} \right)^2 z_2 ; \quad v_b = k \frac{b}{z_2} y_2.
  \]

- According to (256), the diffracted field for Fresnel diffraction through a rectangular aperture is given by
  \[
  u_I = I_xI_y \frac{C'}{i\lambda z_2} ; \quad C' = e^{ik|z_2 + \frac{x_2^2 + y_2^2}{2}}.
  \]
  (264)
  Since
  \[
  \frac{v_a^2}{2u_a} = \frac{kx_2^2}{2z_2} ; \quad \frac{v_b^2}{2u_b} = \frac{ky_2^2}{2z_2} ; \quad a\sqrt{\frac{\pi}{u_a}} b\sqrt{\frac{\pi}{u_b}} = \frac{\lambda z_2}{2},
  \]
  we get
  \[
  u_I = \frac{1}{2\lambda} e^{ikz_2} \left\{ [C(\alpha_a^+) - C(\alpha_a^-)] + i[S(\alpha_a^+) - S(\alpha_a^-)] \right\} \times \left\{ [C(\alpha_b^+) - C(\alpha_b^-)] + i[S(\alpha_b^+) - S(\alpha_b^-)] \right\}.
  \]
  (266)
  The intensity distribution for Fresnel diffraction through a rectangular aperture becomes:
  \[
  I = |u_I|^2
  \]
  (267)
  where \( u_I \) is given in (266).

- For an infinitely large aperture:
  \[
  \alpha_a^\pm \to \pm \infty ; \quad \alpha_b^\pm \to \pm \infty
  \]
  (268)
  and hence
  \[
  u_I = \frac{1}{2\lambda} e^{ikz_2}(1 + i)(1 + i) = \frac{1}{2\lambda} e^{ikz_2}(1 + 2i - 1) = e^{ikz_2}
  \]
  (269)
as expected.

### 12.6 Half-plane

- Consider a plane wave that is normally incident upon a half-plane, as illustrated in Fig. 11.3.

- Then the diffracted field and intensity follow from (266) and (267) by setting
  1. \( a = \infty \), so that \( \alpha_a^+ = \pm \infty \)
  2. \( b \to \infty \) in \( \alpha_b^- \), so that \( \alpha_b^- = -\infty \)
Figure 18: Diffraction of a plane wave that is normally incident upon the half-plane $z = 0, y \geq 0$.

3. $b \to 0$ in $\alpha_b^+$, so that $\alpha_b^+ \to -\sqrt{\frac{kz}{2y^2}} (= A y_2)$.

- Thus we get

$$u_I = \frac{e^{i k z_2}}{2i} \left\{ 1 + i \left\{ C(-A y_2) + \frac{1}{2} + i \left[ S(-A y_2) + \frac{1}{2} \right] \right\} \right\} \quad (270)$$

$$I = \frac{1}{2} \left\{ C(-A y_2) + \frac{1}{2} \right\} \left\{ S(-A y_2) + \frac{1}{2} \right\} \quad (271)$$

where

$$A = \sqrt{\frac{k}{2y^2}} = \sqrt{\frac{\pi}{\lambda z_2}}. \quad (272)$$

- When $y_2 \to +\infty$, $C = S = -\frac{1}{2}$, and hence $u_I = 0, I = 0$.
- When $y_2 \to -\infty$, $C = S = +\frac{1}{2}$, and hence $u_I = \exp(i k z_2), I = 1$.
- When $y_2 = 0$, $C = S = 0$, and hence $u_I = \frac{1}{2} \exp(i k z_2), I = \frac{1}{4}$.
- For large absolute values of the argument $u$ we have

$$C(u) \sim \frac{1}{2} \text{sgn}(u) + \sin(u^2)/\sqrt{2\pi u} \quad (273)$$

$$C(u) \sim \frac{1}{2} \text{sgn}(u) + \cos(u^2)/\sqrt{2\pi u}. \quad (274)$$

- Thus, for $|u| = A |y_2| \gg 1$, the diffracted field becomes

$$u_I = \frac{e^{ikz_2}}{2} \left\{ 1 - \text{sgn}(y_2) + \frac{e^{i(A^2 y_2^2 + \pi/4)}}{\sqrt{\pi A y_2}} \right\} \quad (275)$$

and the corresponding intensity becomes

$$I = \frac{1}{4} \left\{ (1 - \text{sgn}(y_2))^2 + \frac{1}{\pi A^2 y_2^2} + 2(1 - \text{sgn}(y_2)) \frac{\cos(A^2 y_2^2 + \pi/4)}{\sqrt{\pi A y_2}} \right\}. \quad (276)$$

13 Exact solution for diffraction by a half-plane

13.1 Exact solution

Figure 19: Diffraction of a plane wave by the half-plane $z = 0, y \geq 0$. The propagation direction of the incident plane wave forms an angle $\theta_0$ with the positive $y$ axis.

- Let a plane wave be incident upon a half-plane, and let the wave vector of the incident wave be normal to the edge of the half-plane, but not necessarily normal to the half-plane itself (see Fig. 11.4).
- The exact solution for the diffracted field is given by

$$\begin{bmatrix} u^s \\ u^b \end{bmatrix} = F(\xi^s) u^i \mp F(\xi^b) u^r \quad (277)$$

where

$$\begin{bmatrix} \xi^i \\ \xi^r \end{bmatrix} = \pm \sqrt{2ks} \sin \frac{1}{2} (\theta \mp \theta_0). \quad (278)$$
• Here $u^i$ and $u^r$ are respectively the incident and the reflected plane wave:

$$
\begin{align*}
\begin{cases}
  u^i \\
  u^r
\end{cases} = \frac{e^{ikz}}{e^{-ikz}} = e^{ikx}\cos(\theta-\theta_0).
\end{align*}
$$

(279)

• $\theta$ and $\theta_0$ are the angles between the positive y axis and the directions of incidence and observation, respectively (Fig. 11.4).

• The function $F(x)$ in (277) is a generalised, complex Fresnel integral defined as

$$
F(x) = \frac{e^{-it/4}}{\sqrt{\pi}} \int_x^\infty e^{it^2} \, dt.
$$

(280)

• The solutions $u^s$ and $u^b$ apply to “soft” and “hard” boundary conditions:

$$
u^s = 0 \quad \text{for } z = 0 \text{ and } y \geq 0
$$

(281)

$$
\frac{\partial u^b}{\partial z} = 0 \quad \text{for } z = 0 \text{ and } y \geq 0.
$$

(282)

• In both cases the half-plane is a perfect reflector in the sense that the absolute value of the reflection coefficient is equal to 1, i.e. $|R^s| = |R^b| = 1$.

• For “soft” boundary condition we have a phase shift of $\pi$ upon reflection: $R^s = -R^b = -1$.

• We can express $F(x)$ in terms of the real Fresnel integrals:

$$
F(x) = \frac{1}{2} \left[ 1 - C(x) - S(x) + i[C(x) - S(x)] \right].
$$

(283)

• Since $C(0) = S(0) = 0$ and $C(\pm \infty) = S(\pm \infty) = \pm \frac{1}{2}$, it follows that

$$
F(-\infty) = 1; \quad F(0) = \frac{1}{2}; \quad F(\infty) = 0.
$$

(284)

• Further, we have

$$
|F(x)|^2 = \frac{1}{2} \left\{ \left[C(x) - \frac{1}{2} \right]^2 + \left[S(x) - \frac{1}{2} \right]^2 \right\}.
$$

(285)

• The arguments $\xi^i$ and $\xi^r$ of the generalised Fresnel integrals in (277) are called detour parameters.

• For the special case of normal incidence upon the half-plane, $\theta_0 = \pi/2$, we have

$$
(\xi^i)^2 = ks(1 - \sin \theta) = k(s - z)
$$

(286)

$$
(\xi^r)^2 = ks(1 + \sin \theta) = k[s - (-z)].
$$

(287)

• From Fig. 11.5 and (286) and (287) we see that the square of the detour parameter, i.e. $|\xi^i|^2$ (or $|\xi^r|^2$), is equal to the difference between the phase measured along the diffracted ray and the phase measured along the direct incident (or reflected) ray.

• Note that:

$\xi^i > 0$ when the observation point lies in the shadow zone of the incident wave, i.e. when $\theta < \frac{\pi}{2}$.

$\xi^r > 0$ when the observation point lies in the shadow zone of the reflected wave, i.e. when $\theta > \frac{\pi}{2}$.

• The intensity of the diffracted field is given by

$$
\begin{align*}
\begin{cases}
  I^s \\
  I^b
\end{cases} &= \frac{|u^s|^2}{|u^b|^2} = |F(\xi^i)u^i \mp F(\xi^r)u^r|^2 \\
&= |F(\xi^i)|^2 + |F(\xi^r)|^2 \mp 2\text{Re} \left[ F(\xi^i)u^i F^*(\xi^r)(u^r)^* \right].
\end{align*}
$$

(288)

Figure 20: Diffraction of a plane wave that is normally incident upon the half-plane $z = 0, y \geq 0$. The incident wave propagates in the z direction.
13.2 Comparison with the Kirchhoff solution

• Consider now the case in which \( z > 0 \) and \( |\xi^r| >> 1 \). Then \( |F(\xi^r)| << |F(\xi^i)| \), so that we obtain from (288)

\[
\left\{ \begin{array}{l}
I^s \\
I^h
\end{array} \right\} \approx |F(\xi^i)|^2. \tag{289}
\]

• Further, let the observation point be close to the shadow boundary of the incident wave, so that

\[
s = \sqrt{y^2 + z^2} \approx z + \frac{1}{2} \frac{y^2}{z^2}, \tag{290}
\]

\[
(\xi^i)^2 = k(s - z) \approx \frac{k}{z^2} y^2 = A^2 y^2; \quad A = \sqrt{\frac{k}{z^2}}; \quad \xi^i = Ay^2. \tag{291}
\]

• Using (285), \( C(x) = -C(-x) \), and \( S(x) = -S(-x) \), we get

\[
\left\{ \begin{array}{l}
I^s \\
I^h
\end{array} \right\} \approx \frac{1}{2} \left\{ \left( C(-Ay^2) + \frac{1}{2} \right)^2 + \left( S(-Ay^2) + \frac{1}{2} \right)^2 \right\}. \tag{292}
\]

Comparison of the exact intensity in (292) with the corresponding intensity obtained in the Kirchhoff approximation [see (271)], shows that the two results are equal.

• Thus, when the observation point lies near the shadow boundary of the incident wave (\( \theta \approx \frac{\pi}{2} \)) and \( \xi^r = \sqrt{2ks} \sin \frac{\theta}{2}(\theta + \theta_0) >> 1 \) or \( \sqrt{2ks} >> 1 \), the two exact solutions and the approximate Kirchhoff solution give the same intensity.

14 Focusing and imaging

• Consider the imaging system illustrated in Fig. 12.1, where an on-axis object point emits a diverging spherical wave, which is transformed by a lens into a converging spherical wave with focus or image point at \((0, 0, z_1)\).

14.1 Diffracted field in the focal area

• The converging spherical wave passes through an aperture in the plane \( z = 0 \).

• The diffracted field in the focal area of the lens is obtained by using Rayleigh-Sommerfeld’s first diffraction integral and the Kirchhoff approximation. With \( kR_2 >> 1 \), we have

\[
u_I \approx \frac{1}{i\lambda} \int_A u^i \frac{z_2}{R_2} e^{i k R_2} dxdy \tag{293}
\]

where

\[
R_2 = \sqrt{(x - x_2)^2 + (y - y_2)^2 + z_2^2}. \tag{294}
\]

• The field \( u^i \) that is incident upon the aperture in Fig. 12.1, is a converging spherical wave:

\[
u^i = \frac{e^{-ikR_1}}{R_1} \tag{295}
\]

where \( R_1 \) is the distance from the focal point or image point \((x_1, y_1, z_1)\) to the integration point \((x, y, 0)\):

\[
R_1 = \sqrt{(x - x_1)^2 + (y - y_1)^2 + z_1^2}. \tag{296}
\]

• Restricting our attention to paraxial geometries, using the Fresnel approximation, and letting the image point (focus) lie on the \( z \) axis, so that \( x_1 = y_1 = 0 \), we have
\[ u_I \simeq \frac{C}{i\lambda z_1 z_2} \int \int \exp \left\{ -ik \frac{xx_2 + yy_2}{z_2} \right\} \exp \left\{ \frac{i}{2} \left( \frac{1}{z_2} - \frac{1}{z_1} \right) (x^2 + y^2) \right\} dxdy \]

where

\[ C = \exp \left\{ i k [z_2 - z_1 + \frac{x_2^2 + y_2^2}{2z_2}] \right\}. \]

(297)

14.2 Circular aperture

- We introduce polar co-ordinates to obtain from (297)

\[ u_I \simeq \frac{C}{i\lambda z_1 z_2} 2\pi \int_0^1 J_0(v't) \exp \left\{ -i \frac{1}{2} u'^2 \right\} tdt \]

(299)

where

\[ v' = v \frac{z_1}{z_2} ; \quad v = k \frac{a}{z_1} r \]

(300)

\[ u' = u \frac{z_1}{z_2} ; \quad u = k \left( \frac{a}{z_1} \right)^2 \tilde{z} ; \quad \tilde{z} = z_2 - z_1. \]

(301)

14.3 Classical theory

- The classical theory of focusing is based on the assumption that the distance from the aperture to the focus is infinitely large.

- Hence \( z_1/z_2 \simeq 1, u' \simeq u, \) and \( v' \simeq v, \) so that (299) gives

\[ I = |u_I|^2 = I_0 \left| 2 \int_0^1 J_0(vt) e^{-\frac{1}{2} u'^2} tdt \right|^2 \]

(302)

where \( I_0 = \left( \frac{\pi a^2}{\lambda z_1} \right)^2 \) is the intensity in the focal point \( u = v = 0.\)

- According to the classical theory, the diffraction pattern is symmetric about the focal plane:

\[ I(u, v) = I(-u, v) ; \quad u = k \left( \frac{a}{z_1} \right)^2 \tilde{z}. \]

(303)

- Along the axis \( v = 0, \) (302) gives

\[ I(u, 0) = I_0 \left( \frac{\sin(u/4)}{u/4} \right)^2 = I_0 \text{sinc}^2(u/4). \]

(304)

14.4 Focal shift

The assumptions upon which the classical theory is based are not satisfied at low Fresnel numbers \( N, \) defined by

\[ N = \frac{a^2}{\lambda z_1}. \]

- When \( N \gg 1, \) one can see large deviations between observations and results of the classical theory.

- Then we must return to (299), which gives

\[ I(u', v') = |u_I|^2 = I_0 \left( \frac{z_1}{z_2} \right)^2 \left| 2 \int_0^1 J_0(v't) e^{-\frac{1}{2} u'^2} tdt \right|^2. \]

(305)

- From (301) we have

\[ u' = 2\pi N \frac{z_2 - z_1}{z_2} = 2\pi N \left( 1 - \frac{z_1}{z_2} \right) ; \quad \frac{z_1}{z_2} = 1 - \frac{u'}{2\pi N}. \]

(306)

so that (305) becomes

\[ I(u', v') = I_0 \left( 1 - \frac{u'}{2\pi N} \right)^2 \left| 2 \int_0^1 J_0(v't) e^{-\frac{1}{2} u'^2} tdt \right|^2, \]

(307)

- Still we have a kind of symmetry:

\[ \left( 1 + \frac{u'}{2\pi N} \right)^2 I(u', v') = I(-u', v') \left( 1 - \frac{u'}{2\pi N} \right)^2. \]

(308)

- But there is no symmetry about the focal plane \( \tilde{z} = 0, \) since there is a nonlinear relation between \( u' \) and \( \tilde{z}. \)

- When \( \frac{z_2}{z_1} \simeq 1, \) we have approximate symmetry about the focal plane.
14.5 Aberrations

According to the classical theory, the intensity in the focal area of a perfect imaging system is given by (302), which can be written

\[ I = I_0 \frac{1}{\pi^2} \left| \int_0^{2\pi} \int_0^1 e^{-i\left(vt \cos(\phi - \beta) + \frac{1}{2}ut^2\right)} t dt d\phi \right|^2. \]  

(309)

- If the imaging system is not perfect, we introduce an aberration function \( \phi_0(t, \beta) \), which describes the deviations of the converging wave front will have deviations from spherical shape.

- Then the intensity in (309) becomes

\[ I = I_0 \frac{1}{\pi^2} \left| \int_0^{2\pi} \int_0^1 e^{i\left(k\phi_0(t, \phi) - vt \cos(\phi - \beta) - \frac{1}{2}ut^2\right)} t dt d\phi \right|^2. \]  

(310)

Fig.12.2).

- If the object point lies on the optical axis, we only have spherical aberrations of various orders. For first-order spherical aberration the aberration function is given by

\[ \phi_0 = \delta_1 \lambda t^4 \]  

(311)

where \( \delta_1 \) is the deviation of the wave front from spherical shape at the edge of the aperture, measured in wavelengths.

- As the object point moves away from the axis, coma is the first off-axis aberration to appear. First-order coma is given by

\[ \phi_0 = \delta_2 \lambda t^3 \cos \phi, \]  

(312)

- As the object point moves sufficiently far from the axis, astigmatism starts to play a role. For pure first-order astigmatism the aberration function is

\[ \phi_0 = \delta_3 \lambda t^2 \cos^2 \phi. \]  

(313)

15 Radiation problems

15.1 Field radiated by a localised source

- Our task is to determine the field radiated by a given time-harmonic source \( s(r) \), so that the Helmholtz equation becomes

\[ \left( \nabla^2 + k^2 \right) u(r) = s(r). \]  

(314)

- We define a three-dimensional Fourier transform pair:

\[ a(r) = \left( \frac{1}{2\pi} \right)^3 \iiint_{-\infty}^{\infty} A(k) e^{ik \cdot r} d^3 k, \]  

(315)

\[ A(k) = \iiint_{-\infty}^{\infty} a(r) e^{-ik \cdot r} d^3 r. \]  

(316)

- Expressing both \( u(r) \) and \( s(r) \) as Fourier integrals, we find upon substitution in (314)

\[ \iiint_{-\infty}^{\infty} \left[ (-k_z^2 + k^2) U(k) - S(k) \right] e^{ik \cdot r} d^3 k = 0 \]  

(317)

where \( k_z^2 = k_x^2 + k_y^2 + k_z^2 \).

- The uniqueness of Fourier integrals gives

\[ U(k) = \frac{S(k)}{k_z^2 - k^2}. \]  

(318)

- Thus, the Fourier representation of \( u(r) \) becomes [cf. (315)]

\[ u(r) = -\left( \frac{1}{2\pi} \right)^3 \iiint_{-\infty}^{\infty} \frac{S(k)}{k_z^2 - k^2} e^{ik \cdot r} d^3 k. \]  

(319)

- We perform the \( k_z \) integration in (319) by using the calculus of residues and close the contour of integration in the upper half of the complex \( k_z \) plane for \( z > Z \) and in the lower part of the \( k_z \) plane for \( z < -Z \).
• Since the source is confined to the slab $|z| < Z$, the integral along that part of the integration path which lies on a semi-circle with infinite radius in the upper or lower half-plane, will not contribute to the integral over the closed integration path.

• Thus, the $k_z$ integral in (319) is equal to $2\pi i$ times the sum of the residus of the poles in the upper half-plane when $z > Z$ and equal to $-2\pi i$ times the sum of the residues of the poles in the lower half-plane when $z < -Z$.

• Since $S(k)$ is an entire function of $k_z$, the only singularities in (318) are the poles contained in the factor $\frac{1}{k^2 - k^2_z}$.

• Carrying out the $k_z$ integration in the manner just explained, we find

$$u^\pm(r) = -\frac{1}{4\pi} \left( \frac{i}{2\pi} \int_{-\infty}^{\infty} S(k^\pm) e^{i k^\pm \cdot r} dk_x dk_y \right)$$

where

$$k^\pm = k_x \hat{e}_x + k_y \hat{e}_y \pm k_z \hat{e}_z$$

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} \; ; \; \text{Im}(k_z) \geq 0.$$ (321)

• Here $u^+$ represents $u$ in the half-space $z > Z$, and $u^-$ represents $u$ in the half-space $z < -Z$.

### 15.2 Field due to a point source - Green’s function

• Consider the field radiated by a point source located at the origin. Then the source is

$$s(r) = \delta(x)\delta(y)\delta(z)$$

and hence

$$S(k) = \iiint_{-\infty}^{\infty} s(r) e^{-i k \cdot r} d^3 r = \iiint_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z) e^{-i k \cdot r} dx dy dz = 1.$$ (323)

• Equation (320) now gives

$$u^\pm(r) = -\frac{1}{4\pi} \left( \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{i(k_x x + k_y y + k_z z)} \frac{dk_x dk_y}{k_z} \right).$$ (324)

• The expression inside the parenthesis is Weyl’s plane-wave expansion of a spherical wave. Thus

$$u(r) = u^+(r) = u^-(r) = -\frac{1}{4\pi} \frac{e^{ikr}}{r} ; \; r = \sqrt{x^2 + y^2 + z^2}.$$ (325)

• This particular solution is called the Green’s function. For wave propagation in three dimensions we thus have

$$(\nabla^2 + k^2)G(r) = \delta(x)\delta(y)\delta(z)$$ (326)

where

$$G = -\frac{1}{4\pi} \frac{e^{ikr}}{r}.$$ (327)

• In terms of the Green’s function, the field radiated by a source $s(r)$ can be expressed as

$$u(r) = \iiint_{-\infty}^{\infty} s(r') G(r - r') d^3 r'$$ (328)
where
\[ G(r - r') = -\frac{e^{ik|r-r'|}}{4\pi|r-r'|}. \] (329)

• The physical interpretation of (328) is that the field consists of a sum (integral) of point-source solutions \( G(r - r') \) that are weighted by the factor \( s(r) \).

15.3 Two-dimensional wave propagation

• If the source do not vary with \( y \), so that \( s(r) = s(x, z) = s(r_2) \); \( r_2 = x\hat{e}_x + z\hat{e}_z \) (330)
we obtain in the same manner as in the 3D case
\[ u^\pm(r_2) = -\left(\frac{1}{2\pi}\right)^2 2\pi i \int_{-\infty}^{\infty} \frac{S(k_2^\pm) e^{ik_2^\pm r_2} dk_2}{2k_2^\pm} \] (331)
where
\[ k_2^\pm = k_x\hat{e}_x \pm k_z\hat{e}_z ; \quad k_{z2} = \sqrt{k^2 - k_x^2} ; \quad \text{Im}(k_{z2}) \geq 0. \] (332)

• Here the upper sign applies for \( z > Z \) and the lower sign applies for \( z < -Z \).

15.4 Field radiated by a line source - Green’s function

• For a line source located at the origin, the source is given by
\[ s(r_2) = \delta(x)\delta(z) \] (333)
so that
\[ S(k_2) = 1. \] (334)

• Thus, (331) gives
\[ u^+(r_2) = u^-(r_2) = u(r_2) = -\frac{i}{4\pi} \int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_{z2} z)}}{k_{z2}} dk_{z2}. \] (335)

• Using the plane-wave expansion for the zeroth-order Hankel function of the first kind:
\[ H_0^{(1)}(k|r_2|) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_{z2} z)}}{k_{z2}} dk_{z2} \] (336)
we have
\[ u(r_2) = -\frac{i}{4} H_0^{(1)}(k|r_2|) \] (337)
where
\[ G(r_2) = -\frac{i}{4} H_0^{(1)}(k|r_2|) ; \quad |r_2| = \sqrt{x^2 + z^2}. \] (338)

• In terms of the 2D Green’s function, the field radiated by a line source becomes:
\[ u(r_2) = \int_{-\infty}^{\infty} s(r'_2) G(r_2 - r'_2) d^2r'_2 \] (339)
where
\[ G(r_2 - r'_2) = -\frac{i}{4} H_0^{(1)}(k|r_2 - r'_2|). \] (340)

16 Electromagnetic radiation problems

16.1 Field radiated by localised source

• Maxwell’s equations in Gaussian units are given by:
\[ \nabla \cdot \mathbf{D}(r, t) = 4\pi \rho(r, t) \] (341)
\[ \nabla \times \mathbf{E}(r, t) = -\frac{1}{c} \frac{d\mathbf{B}(r, t)}{dt} \] (342)
\[ \nabla \cdot \mathbf{B}(r, t) = 0 \] (343)
\[ \nabla \times \mathbf{B}(r, t) = \frac{\mu}{c} \frac{d\mathbf{D}(r, t)}{dt} + \frac{4\pi \mu}{c} \mathbf{J}(r, t). \] (344)
The total current density consists of two terms:
\[
\mathbf{J}(\mathbf{r}, t) = \mathbf{J}_0(\mathbf{r}, t) + \sigma \mathbf{E}(\mathbf{r}, t),
\]
(345)

In addition we have the continuity equation
\[
\dot{\mathbf{J}}(\mathbf{r}, t) = -\nabla \cdot \mathbf{J}(\mathbf{r}, t) = -\nabla \cdot [\mathbf{J}_0(\mathbf{r}, t) + \sigma \mathbf{E}(\mathbf{r}, t)].
\]
(346)

If the source is time harmonic, so that
\[
\mathbf{J}_0(\mathbf{r}, t) = \text{Re} \left\{ \tilde{\mathbf{J}}_0(\mathbf{r}, \omega) e^{-i\omega t} \right\},
\]
(347)
the radiated field will be time harmonic as well.

Then each scalar component in (341)-(347) can be expressed as follows
\[
a(\mathbf{r}, t) = \text{Re} \left\{ \tilde{a}(\mathbf{r}, \omega) e^{-i\omega t} \right\}.
\]
(348)

Maxwell’s equations (341)-(344) become:
\[
\nabla \cdot \hat{\mathbf{D}}(\mathbf{r}, \omega) = 4\pi \hat{\mathbf{J}}(\mathbf{r}, \omega)
\]
(349)
\[
\nabla \cdot \hat{\mathbf{E}}(\mathbf{r}, \omega) = \frac{i\omega}{c} \hat{\mathbf{B}}(\mathbf{r}, \omega)
\]
(350)
\[
\nabla \cdot \hat{\mathbf{B}}(\mathbf{r}, \omega) = 0
\]
(351)
\[
\nabla \times \hat{\mathbf{B}}(\mathbf{r}, \omega) = -\frac{i\mu \omega}{c} \hat{\mathbf{D}}(\mathbf{r}, \omega) + \frac{4\pi \mu}{c} [\mathbf{J}_0(\mathbf{r}, \omega) + \sigma \mathbf{E}(\mathbf{r}, \omega)].
\]
(352)

As in the scalar case, we introduce Fourier representations:
\[
\hat{\mathbf{A}}(\mathbf{r}, \omega) = \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathbf{A}}(\mathbf{k}, \omega) e^{i\mathbf{k} \cdot \mathbf{r}} d^3 k
\]
(353)
where
\[
\hat{\mathbf{A}}(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathbf{A}}(\mathbf{r}, \omega) e^{-i\mathbf{k} \cdot \mathbf{r}} d^3 r.
\]
(354)

Substituting (353) in (349)-(352), we obtain algebraic equations for \(\hat{\mathbf{E}}(\mathbf{k}, \omega)\) and \(\hat{\mathbf{B}}(\mathbf{k}, \omega)\).

We solve these algebraic equations, substitute the results in (353), and carry out the \(k_z\) integration in the same way as in the scalar case to obtain:
\[
\hat{\mathbf{E}}^\pm(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \hat{\mathbf{E}}(\mathbf{k}^\pm, \omega) e^{i\mathbf{k}^\pm \cdot \mathbf{r}} d^2 k_y dk_z
\]
(355)
\[
\hat{\mathbf{B}}^\pm(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \hat{\mathbf{B}}(\mathbf{k}^\pm, \omega) e^{i\mathbf{k}^\pm \cdot \mathbf{r}} d^2 k_y dk_z
\]
(357)
\[
\mathbf{B}(\mathbf{k}^\pm, \omega) = \frac{c}{\omega} \mathbf{k}^\pm \times \mathbf{E}(\mathbf{k}^\pm, \omega) = -\frac{\mu (\mathbf{k}^\pm \times \tilde{\mathbf{J}}_0(\mathbf{k}^\pm))}{2\pi c k_z}.
\]
(358)

16.2 Field radiated by a dipole

Let the source be a dipole located at the origin and polarised along the unit vector \(\hat{n}\). Then we have
\[
\mathbf{J}_0(\mathbf{r}, t) = \Re \left\{ \tilde{\mathbf{J}}_0(\mathbf{r}, \omega) e^{-i\omega t} \right\} ; \quad \tilde{\mathbf{J}}_0(\mathbf{r}, \omega) = \hat{n} I \delta(\mathbf{r})
\]
(359)
so that
\[
\tilde{\mathbf{J}}_0(\mathbf{k}) = \hat{n} I \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d^2 r = \hat{n} I
\]
(360)
where \(I\) is the dipole strength.

Thus, from (355) - (358):
\[
\mathbf{E}^\pm = \frac{\omega \mu I}{2\pi c^2 k_z} \int_{-\infty}^{\infty} \frac{\mathbf{k}^\pm \times (\mathbf{k}^\pm \times \hat{n})}{k_z} e^{i\mathbf{k}^\pm \cdot \mathbf{r}} d^2 k_y dk_z
\]
(361)
\[
\mathbf{B}^\pm = -\frac{\mu I}{2\pi c} \int_{-\infty}^{\infty} \frac{\mathbf{k}^\pm \times \hat{n}}{k_z} e^{i\mathbf{k}^\pm \cdot \mathbf{r}} d^2 k_y dk_z.
\]
(362)
Using Weyl's plane-wave expansion of a spherical wave, we can rewrite these expressions as:

\[ E = \frac{i\omega \mu I}{c^2 k^2} \nabla \times \nabla \times \left[ \hat{n} \frac{e^{ikr}}{r} \right] \quad (363) \]

\[ B = \frac{\mu I}{c} \nabla \times \left( \hat{n} \frac{e^{ikr}}{r} \right). \quad (364) \]

Carrying out the differentiations, the expressions for \( E \) and \( B \) become

\[ B = -i \frac{k \mu I}{c} e^{ikr} \left( 1 + \frac{i}{kr} \right) \hat{n} \times \hat{e}_r \quad (365) \]

\[ E = -i \frac{\omega \mu I}{c^2} e^{ikr} \left\{ \left( 1 + \frac{3i}{kr} - \frac{3}{(kr)^2} \right) \hat{e}_r (\hat{e}_r \cdot \hat{n}) - \left( 1 + \frac{i}{kr} - \frac{1}{(kr)^2} \right) \hat{n} \right\}. \quad (366) \]

Figure 23: Co-ordinate systems related to the study of the field radiated by a dipole that is placed at the origin and polarized along the \( z \) axis.

With a co-ordinate system such that \( \hat{n} \) points along the \( z \) axis, and with spherical co-ordinates \( r, \theta, \) and \( \phi \), as shown in Figure 13.2, we have

\[ E = E_r \hat{e}_r + E_\theta \hat{e}_\theta; \quad B = B_\phi \hat{e}_\phi, \quad (367) \]

When \( kr \gg 1 \), we may neglect \( E_r \) and the higher-order terms in \( E_\theta \) and \( B_\phi \) to obtain

\[ E_\theta \sim -i \frac{\omega \mu I}{c^2} e^{ikr} \sin \theta \quad (368) \]

\[ B_\phi \sim -i \frac{k \mu I}{c} e^{ikr} \sin \theta. \quad (369) \]

In vacuum, where \( \mu = \varepsilon = 1 \), \( \sigma = 0 \), and \( k = \frac{\omega}{c} \), the result in the far zone becomes

\[ E_\theta \sim B_\phi \sim -i \frac{\omega I}{c} \left( \frac{e^{ikr}}{r} \right) \sin \theta. \quad (370) \]

Thus, in the far zone \( E \) and \( B \) are of equal size, and they are normal to one another and to \( \hat{e}_r \), which now points in the direction of the Poynting vector.

This implies that the far field radiated by a dipole behaves locally as a plane wave.

Note, however, that the amplitude of the field is proportional to \( \sin \theta \), which means that the radiated energy is proportional to \( \sin^2 \theta \). Therefore, the dipole does not radiate energy along its own axis.

17 Retarded solution of the wave equation

Consider the inhomogeneous, scalar wave equation in a uniform, non-dispersive medium:

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \tilde{u}(r, t) = \tilde{s}(r, t). \quad (371) \]

By Fourier decomposition of the field, we have

\[ \tilde{u}(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(r, \omega) e^{-i\omega t} d\omega = 2\text{Re} \frac{1}{2\pi} \int_{0}^{\infty} u(r, \omega) e^{-i\omega t} d\omega \quad (372) \]

which implies that \( u(r, \omega) \) satisfies the inhomogeneous Helmholtz equation:

\[ \left( \nabla^2 + k^2 \right) u(r, \omega) = s(r, \omega) \quad ; \quad k = \frac{\omega}{c}. \quad (373) \]

The solution of (373) can be written

\[ u(r, \omega) = \iint_{-\infty}^{\infty} s(r, \omega) G(r - r', \omega) d^3r' \quad ; \quad G(r - r', \omega) = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{-4\pi |\mathbf{r} - \mathbf{r}'|}. \quad (374) \]
• On substituting (374) in (372), we obtain:

\[ u(r, t) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{s(r', t - \frac{1}{c}||r - r'||)}{r - r'} d^3r'. \] (375)

• This expression is called the *retarded* solution of the wave equation.

• The physical interpretation of this solution is that the field at the observation point \( r \) at time \( t \) consists of a sum of contributions from various source elements \( r' \) that are radiated at the preceding time \( t - \frac{1}{c}||r - r'|| \).

• Each contribution has to leave the source element at this earlier time in order to reach the observation point at the given time \( t \).

\[ \text{Figure 24: Geometry for interpretation of the retarded solution of the wave equation.} \]

18 Asymptotic diffraction theory

• According to Huygen’s principle, the field that is diffracted through an aperture consists of contributions from an infinite number of secondary waves, one from each point in the aperture. Mathematically this sum is expressed as a diffraction integral.

• By evaluating the diffraction integral by means of asymptotic techniques, one can show that only a few of the secondary waves contribute significantly to the diffracted field.

\[ \text{Figure 25: Illustration of a two-dimensional boundary-value problem where the field in the plane } z = 0 \text{ is known and the field in the half-space } z > 0 \text{ is to be determined.} \]

18.1 Two-dimensional and three-dimensional diffraction problems

• Consider three-dimensional (3D) and two-dimensional (2D) wave propagation in a homogeneous medium, and let the field be generated by sources in the half-space \( z \leq 0 \), and let it be known in the plane \( z = 0 \) (cf. Figure 14.1).

• In the Kirchhoff approximation the solution to 3D and 2D diffraction problems can be expressed as

\[ \text{3D: } u_I(x_2, y_2, z_2) = \int_A g(x, y) e^{ikf(x, y)} dx dy \] (376)

\[ f(x, y) = R_1 + R_2 ; \quad g(x, y) = \frac{1}{i\lambda} \frac{z_2}{R_1 R_2} \frac{1}{R_1 R_2} \] (377)

\[ R_j = \sqrt{(x - x_j)^2 + (y - y_j)^2 + z_j^2}. \] (378)

\[ \text{2D: } u_I(x_2, z_2) = \int_a^b g(x) e^{ikf(x)} dx, \] (379)

\[ f(x) = R_1 + R_2 ; \quad g(x) = \frac{1}{i\lambda} \frac{1}{\sqrt{R_1 R_2}} \] (380)

\[ R_j = \sqrt{(x - x_j)^2 + z_j^2}. \] (381)
18.2 The method of stationary phase for single integrals

- For simplicity we concentrate on two-dimensional diffraction problems, so that the diffraction field can be expressed as:

\[ J = \int_{x_1}^{x_2} g(x) e^{ikf(x)} \, dx. \]  
(382)

- When \( k|x_2 - x_1| \) is sufficiently large, then \( \exp[ikf(x)] \) will oscillate so rapidly compared to \( g(x) \) that cancellation occurs except in the immediate neighbourhood of stationary points \( x = x_s \), where \( f'(x_s) = 0 \), or in the immediate neighbourhood of either end point \( x = x_1 \) or \( x = x_2 \).

- For simplicity we assume that we have isolated stationary points and end points.

Figure 26: Diffraction of a spherical wave originating at the point \((x_1, y_1, z_1)\) through an aperture \( A \) in the plane \( z = 0 \). The diffracted field is observed in \((x_2, y_2, z_2)\).

Isolated, interior stationary point Suppose we have an isolated, interior stationary point \( x_s \), so that \( x_1 << x_s << x_2 \).

- In order to determine the asymptotic contribution to the integral \( J \) in (382) we expand \( f(x) \) and \( g(x) \) about \( x = x_s \), so that

\[ g(x) = g_0 + g_1 t + g_2 t^2 + \ldots \]
\( f(x) = f_0 + f_2 t^2 + f_3 t^3 + \ldots \)

where

\[ g_n = \frac{1}{n!} \left. \frac{\partial^n g}{\partial x^n} \right|_{x = x_s} \quad ; \quad f_n = \frac{1}{n!} \left. \frac{\partial^n f}{\partial x^n} \right|_{x = x_s} \]  
(384)

- Further, we write

\[ e^{ikf(x)} = e^{ikf_0 + ikf_2 t^2} e^{ik\Delta f} \quad ; \quad \Delta f = f_3 t^3 + f_4 t^4 \ldots \]  
(385)

and we expand \( e^{ik\Delta f} \):

\[ e^{ik\Delta f} = 1 + ik\Delta f + \frac{1}{2} (ik\Delta f)^2 + \ldots . \]  
(386)

- To the lowest asymptotic order we get the following contribution \( J_S \) to the integral \( J \) in (382) from the interior stationary point \( x_s \):

\[ J_S = \sqrt{\frac{n}{k|f_2|}} g_0 e^{[k f_0 + \frac{\pi}{4} \text{sgn}(f_2)]} \]  
(387)

- Higher-order terms in the asymptotic contribution from an isolated, interior stationary point are given by (cf. equations (8.8a)-(8.8g) in Stamnes, 1986)

\[ J_S \sim \sqrt{\frac{n}{k|f_2|}} e^{[k f_0 + \frac{\pi}{4} - \frac{\pi}{4} \text{arg}(f_2)] [Q_0 + Q_2 + Q_4]} \]  
(388)

where

\[ Q_0 = g_0 \]  
(389)

\[ Q_2 = \frac{i}{k|f_2|} \left( \frac{3}{4} f_3 + \frac{3}{8} g_0 f_3 + \frac{15}{16} g_0 f_3^2 \right) \]  
(390)

\[ Q_4 = \frac{1}{(k|f_2|)^2} \left[ - \frac{3}{4} f_4 + \frac{15}{8} f_2 f_4 + \frac{15}{32} f_2^2 f_4 + \frac{315}{64} f_2^3 f_4 - \frac{3465}{512} g_0 \left( \frac{f_3}{f_2} \right)^4 \right] . \]  
(391)

- Here the coefficients \( A, B, \) and \( C \) are given by

\[ A = g_3 f_3 + g_2 f_4 + g_1 f_5 + g_0 f_6 \]  
(392)

\[ B = g_3 f_2^3 + 2 g_1 f_3 f_4 + g_0 (f_2^2 + 2 f_3 f_5) \]  
(393)

\[ C = g_1 f_2^3 + 3 g_0 f_2 f_4^2 f_3. \]  
(394)
**Stationary end point**  Let the stationary point coincide with the lower end point, so that $x_s = x_1$.

- Then we get the following asymptotic contribution

$$J_{SE} \sim J_{even} + J_{odd} \sim \frac{1}{2} \left( \frac{\pi}{|f_2|} \right)^{1/2} e^{i(kf_0 + \psi)} (Q_0 + Q_1 + Q_2 + Q_3 + Q_4)$$  (395)

where

$$\psi = \frac{\pi}{4} - \frac{1}{2} \arg f_2.$$  (396)

- Here $Q_1$ and $Q_3$ are given in equations (8.11b) and (8.11c) in Stamnes (1986).

- Note that to the lowest asymptotic order the contribution from an end point is half as large as that from an interior stationary point.

**Non-stationary end point**  To determine the asymptotic contribution from a non-stationary end point we may follow a similar procedure as in the previous two cases.

- But in this case it is just as simple to use integration by parts. Thus, we

- To the lowest asymptotic order the contribution from a non-stationary end point $x = x_j$ ($j = 1, 2$) becomes

$$J_E = (-1)^j \frac{g(x_j)}{ikf'(x_j)} e^{ikf(x_j)},$$  (397)

where $j = 2$ applies to an upper end point, and where $j = 1$ applies to a lower end point.