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1 Vector relations

Show that

\[ \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}. \]  

(1)

2 Change of variables in the wave equation

In section 4.1 in the lecture notes [5] the equation

\[ \frac{\partial^2 V}{\partial p \partial q} = 0 \]  

(2)

constitutes an important part of the derivation of a general solution of the wave equation.

Show that

\[ \frac{\partial^2 V}{\partial \zeta^2} - \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2} = 0 \]  

(3)

can be expressed in the form:

\[ \frac{\partial^2 V}{\partial p \partial q} = 0 \]  

(4)

where \( p = \zeta - vt \) og \( q = \zeta + vt \).

3 Laplacian operator for spherically symmetric functions

Show that the Laplacian operator has the following form in spherical coordinates (used in section 4.2 in the lecture notes [5]):

\[ \nabla^2 f(r) = \frac{1}{r} \frac{\partial}{\partial r} \left[ r f(r) \right] \]  

(5)

where

\[ r = \sqrt{x^2 + y^2 + z^2}. \]  

(6)

4 Waves and wave packets

4.1 Superposition of two harmonic plane waves

Two harmonic plane waves of equal amplitudes propagate in the positive \( z \) direction. One of the waves has frequency \( \omega_1 \) and wave number \( k_1 \), and the other wave has frequency \( \omega_2 \) and wave number \( k_2 \).

Find a formula for the sum of the two waves expressed in terms of the
• Average frequency: $\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2)$,
• Average wave number: $\bar{k} = \frac{1}{2}(k_1 + k_2)$
• Difference frequency: $\Delta \omega = \omega_1 - \omega_2$ and
• Difference wave number: $\Delta k = k_1 - k_2$.

4.2 Group velocity and phase velocity

Show that when $\Delta \omega/\bar{\omega} << 1$, the sum of the two waves can be interpreted as a harmonic plane wave that propagates in the positive $z$ direction with phase velocity $v = \bar{\omega}/\bar{k}$ and with a slowly varying amplitude that propagates in the positive $z$ direction with group velocity $v_g = \Delta \omega/\Delta k$. Sketch the sum of the two waves.

4.3 Dispersion and energy propagation

Show from the results in Exercise 4.2 that i) $v_g \neq v$ in a dispersive medium. ii) $v_g = v$ in a non-dispersive medium.

5 Connection between group velocity, phase velocity, and refractive index

5.1 Group velocity as a function of phase velocity and refractive index

Show that the group velocity $v_g$ can be expressed in terms of the phase velocity $v$ and the refractive index $n(\omega)$ in the following manner:

$$\frac{1}{v_g} = \frac{1}{v} + \frac{\omega}{c} \frac{dn(\omega)}{d\omega}. \quad (7)$$

5.2 The group velocity is less than the speed of light in vacuum

Given that

$$n = \sqrt{1 - \frac{B}{\omega^2 - \omega_0^2}}; \quad B > 0, \quad (8)$$

show that $v_g < c$ when $\omega < \omega_0$. 
6 Propagation in a dispersive medium

Consider a polychromatic plane wave that propagates in the $z$ direction in a dispersive medium, so that

$$u(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega) e^{i[k(\omega)z-\omega t]} d\omega. \quad (9)$$

It follows from (9) that in the plane $z = 0$

$$u(0, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega) e^{-i\omega t} d\omega = g(t) \quad (10)$$

so that

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} u(0, t) e^{i\omega t} dt \quad (11)$$

where $\tilde{g}(\omega)$ is the frequency spectrum of the plane wave in the plane $z = 0$. Let the frequency spectrum $\tilde{g}(\omega)$ have its maximum value at $\omega = \omega_0$, and let $\tilde{g}(\omega)$ fall off rapidly from this value, so that we may represent $k(\omega)$ by the first two terms in a Taylor series around $\omega_0$:

$$k(\omega) = k(\omega_0) + \left. \frac{dk}{d\omega} \right|_{\omega=\omega_0} (\omega - \omega_0). \quad (12)$$

6.1 Propagation of a quasi-monochromatic wave

Show that when we neglect terms of higher order than those retained in (12), we can express $u(z, t)$ in (9) as follows:

$$u(z, t) \approx e^{i\omega_0 z} \left( \frac{1}{v_0} - \frac{1}{v_{g0}} \right) g \left( -\frac{z}{v_{g0}} + t \right) \quad (13)$$

where

$$v_0 = \frac{\omega_0}{k(\omega_0)} \quad ; \quad v_{g0} = \left. \frac{d\omega}{dk} \right|_{\omega=\omega_0}. \quad (14)$$

6.2 The shape and speed of the wave

Give a physical interpretation of the result in 6.1.

6.3 Alternative way to proceed

Show that the same result as in (13) can be obtained by considering

$$u(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega) e^{i\frac{z}{v_{g0}} f(\omega)} d\omega \quad (15)$$
where

\[ f(\omega) = \omega[n(\omega) - \theta]; \quad \theta = \frac{ct}{\varepsilon}, \]  

(16)
and expanding \( f(\omega) \) in a Taylor series around \( \omega_0 \) to the first order, i.e.

\[ f(\omega) = f(\omega_0) + f'(\omega_0)(\omega - \omega_0). \]  

(17)

7 Polarisation and rotation of co-ordinate system

7.1 Maximum and minimum values

Show that the ellipse (Eq. (15), page 25 in Born and Wolf [2]):

\[ \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - 2\frac{x}{a_1 a_2} \cos \delta = \sin^2 \delta \]  

(18)
has the following maximum and minimum values

\[ y_{\text{max}} = a_2 \quad \text{for} \quad x = a_1 \cos \delta \]
\[ y_{\text{min}} = -a_2 \quad \text{for} \quad x = -a_1 \cos \delta \]
\[ x_{\text{max}} = a_1 \quad \text{for} \quad y = a_2 \cos \delta \]
\[ x_{\text{min}} = -a_1 \quad \text{for} \quad y = -a_1 \cos \delta. \]  

(19)

7.2 Rotation of co-ordinate system

Show that by rotating the co-ordinate system an angle \( \psi \), so that

\[ x = \xi \cos \psi - \eta \sin \psi \quad ; \quad y = \xi \sin \psi + \eta \cos \psi \]  

(20)
the coefficients in front of the \( \xi \eta \) term in (18) will disappear if \( \psi \) satisfies the relation

\[ \tan(2\psi) = \tan(2\alpha) \cos \delta \quad ; \quad \tan \alpha = \frac{a_2}{a_1}. \]  

(21)

7.3 Simplification of the expression for the ellipse

Show that in \((\xi, \eta)\) co-ordinates (18) can be written as follows:

\[ b^2\xi^2 + a^2\eta^2 = a^2b^2 \]  

(22)
where

\[ a^2 = a_2^2 \sin^2 \psi + a_1^2 \cos^2 \psi - 2a_1 a_2 \cos \delta \sin \psi \cos \psi \]
\[ b^2 = a_2^2 \cos^2 \psi + a_1^2 \sin^2 \psi + 2a_1 a_2 \cos \delta \sin \psi \cos \psi. \]  

(23)
8 Phase velocity and group velocity for surface waves on water

The propagation of linear, harmonic surface waves in water of constant depth is governed by the equations

\[
\nabla^2 \phi(x, y, z) = 0 ; \quad d < y < 0 \tag{24}
\]

\[-\omega^2 \phi(x, y, z) + g \frac{\partial \phi(x, y, z)}{\partial y} = 0 ; \quad y = 0 \tag{25}
\]

\[\frac{\partial \phi(x, y, z)}{\partial y} = 0 ; \quad y = -d \tag{26}
\]

\[\eta(x, z) = \frac{i\omega}{g} \phi(x, 0, z). \tag{27}\]

The symbols in the equations above have the following meaning: \(\omega = 2\pi/T\); \(T\) is the period, \(d =\) water depth, \(\phi =\) velocity potential, \(g =\) acceleration of gravity, and \(\eta =\) displacement of the water surface from the position \(y = 0\), which is its position at rest. The velocity \(\mathbf{v}\) of a water "particle" is given as

\[\mathbf{v} = \nabla \phi. \tag{28}\]
8.1 Separation of variables

Use separation of variables to find the solution of (24). Thus, express $\phi$ as a product

$$\phi = A(y)B(x, z)$$  \hspace{1cm} (29)

where $A$ only depends on $y$ and $B$ only depends on $x$ and $z$, and show by substitution into (24) that

$$\frac{\partial^2 A(y)}{\partial^2 y} - k^2 A(y) = 0$$ \hspace{1cm} (30)

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) B(x, z) = 0$$ \hspace{1cm} (31)

where $k^2$ is a separation constant.

8.2 Dispersion relation

Show that

$$A(y) = C \cosh[k(y + d)]$$ \hspace{1cm} (32)

where $C$ is a constant, satisfies (26) and (30). Substitute this solution into (25) and show that the dispersion relation, i.e. the relation between $\omega$ and $k$ is as follows:

$$\omega^2 = gk \tanh(kd).$$  \hspace{1cm} (33)

8.3 Phase velocity and group velocity

Find the phase velocity and the group velocity.
8.4 The phase velocity of surface waves in water of infinite depth

Show that when the water depth increases, so that $kd \to \infty$, then the phase velocity approaches the following limiting value

$$v \to v_0 = \frac{g}{2\pi}T$$

(34)

whereas the group velocity approaches the limiting value

$$v_g \to v_{g0} = \frac{1}{2}v_0.$$  

(35)

8.5 The phase velocity of surface waves in water of finite depth

Show that $v_g < v$ also when the depth is finite.

8.6 The refractive index

The refractive index for water waves is defined as

$$n = \frac{v_0}{v}$$

(36)

where $v_0$ is the phase velocity in water of infinite depth given by (34). Note the similarity with light waves, in which case $v_0$ corresponds to the speed of light in vacuum. Show that $n$ can be expressed as

$$n = \coth(nk_0d)$$

(37)

where

$$k_0 = \frac{\omega}{v_0} = \frac{\omega^2}{g} = \frac{4\pi^2}{gT^2}$$

(38)

(Hint: Use that $k = \frac{\omega}{v} = \frac{\omega}{v_0} = \frac{1}{2}v_0$.)

Determine $n$ numerically for $T = 12$ s and $d = 100$ m and for $T = 12$ s and $d = 25$ m.

8.7 The wavelength in water of infinite depth

Find an expression for the wavelength $\lambda_0$ in deep water for a harmonic plane surface wave with period $T$. Determine $\lambda_0$ when $T = 10$ s and $T = 15$ s.
8.8 The wavelength in water of finite depth

Determine the wavelength $\lambda$ for the wave in water of constant depth $d$ expressed in terms of $\lambda_0$, $d$, and the refractive index $n$. Compute $\lambda$ for $T = 12$ s and $d = 100$ m and for $T = 12$ s and $d = 25$ m.

8.9 Refraction of waves that propagate towards the shore

A plane wave with a period $T = 12$ s propagates from infinitely deep water towards an area with a constant finite depth of $d = 25$ m. The angle of incidence $\theta^i$ (see Fig. 3) is $30^\circ$. Use Snell’s law for water waves ($n_1 \sin \theta^i = n_2 \sin \theta^f$) to determine the angle of refraction $\theta^f$. 
9 Fresnel’s formulas

9.1 Rewriting of Fresnel’s formulas

Use Snell’s law to show that the Fresnel formulas for reflection and transmission

\[ T_{TM} = \frac{2n_1 \cos \theta^i}{n_2 \cos \theta_i + n_1 \cos \theta^i} \quad ; \quad R_{TM} = \frac{n_2 \cos \theta^i - n_1 \cos \theta^i}{n_2 \cos \theta^i + n_1 \cos \theta^i} \]

\[ T_{TE} = \frac{2n_1 \cos \theta^i}{n_1 \cos \theta^i + n_2 \cos \theta^i} \quad ; \quad R_{TE} = \frac{n_1 \cos \theta^i - n_2 \cos \theta^i}{n_1 \cos \theta^i + n_2 \cos \theta^i} \] (39)

can be expressed as follows:

\[ T_{TM} = \frac{2 \sin \theta^i \cos \theta^i}{\sin(\theta^i + \theta^i) \cos(\theta^i - \theta^i)} \quad ; \quad R_{TM} = \frac{\tan(\theta^i - \theta^i)}{\tan(\theta^i + \theta^i)} \]

\[ T_{TE} = \frac{2 \sin \theta^i \cos \theta^i}{\sin(\theta^i + \theta^i)} \quad ; \quad R_{TE} = -\frac{\sin(\theta^i - \theta^i)}{\sin(\theta^i + \theta^i)} \] (40)

9.2 The sign of the reflection and transmission coefficients

Provided that \( \theta^i \) and \( \theta^t \) are real angles, determine under what circumstances \( T_{TM} \), \( R_{TM} \), \( T_{TE} \), and \( R_{TE} \) are positive and negative. What physical interpretation do we associate with negative values?

10 Reflectivity and transmissivity

10.1 Energy conservation for \( TE \) and \( TM \) components

From the formulas for reflectivity and transmissivity

\[ T_{TM} = \frac{\sin 2\theta^i \sin 2\theta^t}{\sin^2(\theta^i + \theta^t) \cos^2(\theta^i - \theta^t)} \quad ; \quad R_{TM} = \frac{\tan^2(\theta^i - \theta^t)}{\tan^2(\theta^i + \theta^t)} \]

\[ T_{TE} = \frac{\sin 2\theta^i \sin 2\theta^t}{\sin^2(\theta^i + \theta^t)} \quad ; \quad R_{TE} = -\frac{\sin(\theta^i - \theta^t)}{\sin(\theta^i + \theta^t)} \] (41)

show that

\[ T_{TM} + R_{TM} = 1 \quad ; \quad T_{TE} + R_{TE} = 1. \] (42)
10.2 Energy conservation

Show that
\[ T + R = 1 \]  
(43)

where
\[ R = R^{TM} \cos^2 \alpha_i + R^{TE} \sin^2 \alpha_i \]
\[ T = T^{TM} \cos^2 \alpha_i + T^{TE} \sin^2 \alpha_i. \]  
(44)

11 Total reflection 1

Consider a harmonic, plane wave that is incident upon a plane interface under an angle of incidence \( \theta_i \) that is greater than the critical angle \( \theta_{ic} \), so that we have total reflection. The components of the electric field are then given by (see section 9.3.5 of [5]):

\[
E_x = T^{TM} E^{TMi} \cos \theta_i \ e^{i(k_x x - \omega t)} e^{-|k_z| z}
\]
\[
E_y = T^{TE} E^{TEi} \ e^{i(k_x x - \omega t)} e^{-|k_z| z}
\]
\[
E_z = -\frac{\sin \theta_i}{n} T^{TM} E^{TMi} \ e^{i(k_x x - \omega t)} e^{-|k_z| z}
\]  
(45)

where
\[
k_z = k_1 \sin \theta_i = \frac{1}{n} \frac{\omega}{n_2} \sin \theta_i \quad ; \quad n = \frac{n_2}{n_1} < 1
\]
\[
k_{z2} = k_2 \cos \theta_i \quad ; \quad \cos \theta_i = \frac{i}{n} \sqrt{\sin^2 \theta_i - n^2}.
\]  
(46)
11.1 Transmitted magnetic field

Show from Maxwell’s equation (with $\mu_1 = \mu_2 = 1$)
\[ \nabla \times \mathbf{E}^t = -\frac{1}{c} \mathbf{B}^t \] (47)
that the transmitted magnetic field $\mathbf{H}^t$ has the following components:
\[
\begin{align*}
H_x^t &= -n_2 \cos \theta T^{TE} E^{TEi} e^{i(k_x x - \omega t)} e^{-|k_z|z} \\
H_y^t &= n_2 T^{TM} E^{TMi} e^{i(k_x x - \omega t)} e^{-|k_z|z} \\
H_z^t &= n_1 \sin \theta T^{TE} E^{TEi} e^{i(k_x x - \omega t)} e^{-|k_z|z}.
\end{align*}
\] (48)

11.2 Transmitted Poynting vector

Show that the Poynting vector of the transmitted field
\[ \mathbf{S}^t = \frac{c}{4\pi} \mathbf{E}^t \times \mathbf{H}^t \] (49)
has the following components:
\[
\begin{align*}
S_x^t &= \frac{c_1 n_1 \sin \theta}{16\pi} e^{-2A} \left( (E^{TEi})^2 \left[ (T^{TE})^2 e^{2i\phi} + ((T^{TE})^*)^2 e^{-2i\phi} + 2|T^{TE}|^2 \right] \\
&\quad + (E^{TMi})^2 \left[ (T^{TM})^2 e^{2i\phi} + ((T^{TM})^*)^2 e^{-2i\phi} + 2|T^{TM}|^2 \right] \right) \\
S_y^t &= \frac{c_1 n_1 \sin \theta \cos \theta}{8\pi} e^{-2A} \frac{E^{TMi} E^{TEi}}{T^{TE} \left[ (T^{TM})^* T^{TE} - T^{TM} (T^{TE})^* \right]} \left[ (T^{TM})^2 e^{2i\phi} + ((T^{TM})^*)^2 e^{-2i\phi} \right] \\
&\quad + (E^{TEi})^2 \left[ (T^{TE})^2 e^{2i\phi} + ((T^{TE})^*)^2 e^{-2i\phi} \right] \\
S_z^t &= \frac{cn_2 \cos \theta}{16\pi} e^{-2A} \left( (E^{TMi})^2 \left[ (T^{TM})^2 e^{2i\phi} + ((T^{TM})^*)^2 e^{-2i\phi} \right] \\
&\quad + (E^{TEi})^2 \left[ (T^{TE})^2 e^{2i\phi} + ((T^{TE})^*)^2 e^{-2i\phi} \right] \right)
\end{align*}
\] (50)

where
\[ A = |k_z|z \quad ; \quad \phi = k_x x - \omega t. \] (51)

11.3 Time average of the Poynting vector

Show that the time-average of the $z$ component of the Poynting vector vanishes, i.e.
\[ < S_z^t > = 0, \] (52)
and that the time-average of the $x$ and $y$ components are given by:
\[
\begin{align*}
< S_x^t > &= \frac{c_1 n_1 \sin \theta_i}{4\pi} e^{-2a} |(T^{TE})^2 (E^{TEi})^2 + |T^{TM}|^2 (E^{TEi})^2 | \cos \theta_i | \left( (T^{TM})^* T^{TE} E^{TMi} E^{TEi} \right) \\
< S_y^t > &= \frac{c_1 n_1 \sin \theta_i}{4\pi} e^{-2a} |(T^{TM})^* T^{TE} E^{TMi} E^{TEi} | \left( (T^{TE})^2 (E^{TEi})^2 + |T^{TM}|^2 (E^{TEi})^2 \right)
\end{align*}
\] (53)

What is the physical explanation of this result? (Time averaging implies integration over an interval $T'$ that is much larger than the period $T = \frac{2\pi}{\omega}$, i.e. $< S^t > = \frac{1}{2\pi} \int_{T'}^{T'} S^t dt$, where $T' >> T$.)
12 Fresnel’s rhomb

Fig. 6 shows a Fresnel’s rhomb, which can be used to produce circularly polarised light from linearly polarised light or vice versa. The required phase difference of $\delta = 90^\circ$ can be obtained through two successive total reflections, each introducing a phase difference of $45^\circ$. For a single total reflection the phase difference $\delta$ is given by the formula:

$$\tan \frac{\delta}{2} = \frac{\cos \theta^i \sqrt{\sin^2 \theta^i - n^2}}{\sin^2 \theta^i}$$

(54)

where $\theta^i$ is the angle of incidence and $n = \frac{n_2}{n_1} < 1$.

12.1 $\sin \theta^i$

Solve (54) with respect to $\sin \theta^i$, and show that

$$\sin \theta^i = \left( \frac{n^2 + 1 \pm \sqrt{(n^2 + 1)^2 - 4n^2(1 + \tan^2 \frac{\delta}{2})}}{2(\tan^2 \frac{\delta}{2} + 1)} \right)^{\frac{1}{2}}.$$  

(55)

12.2 Phase difference of $45^\circ$, $n = 1.52$

For $n_{21} = \frac{1}{n} = 1.52$ determine those angles of incidence which give a phase difference of $45^\circ$.

12.3 Phase difference of $45^\circ$, $n = 1.49$

Repeat the task in exercise 12.2 for $n_{21} = 1.49$, and explain the result.
12.4 Maximum phase difference

Show from (54) that $\delta$ has a maximum value $\delta_m$ for $\theta_i = \theta_{im}$ given by

$$\sin^2 \theta_{im} = \frac{2n^2}{1 + n^2}$$

and that $\delta_m$ is given by:

$$\tan \frac{\delta_m}{2} = \frac{1 - n^2}{2n}.$$  \hspace{1cm} (57)

12.5 Phase differences of $45^\circ$ and $90^\circ$

What minimum value of $n_{12} = \frac{1}{n}$ is required to obtain a phase difference of

1. $90^\circ$,
2. $45^\circ$?

13 Total reflection 2

Consider reflection and refraction of a plane wave at a plane interface between two media, and let $n_2 \leq n_1$, as illustrated in Fig. 7. From Snell’s law

$$n_1 \sin \theta^i = n_2 \sin \theta^i$$

(58)
or
\[ \sin \theta_t = \frac{\sin \theta_i}{n}; \quad n = \frac{n_2}{n_1} \leq 1 \quad (59) \]

it follows that when \( \theta^i > \theta^{ic} \), where \( \sin \theta^{ic} = n \), then \( \sin \theta_t > 1 \). Thus, \( \theta_t \) must be a complex number. Show that \( \theta_t = \alpha + i\beta \), where

\[ \alpha = \frac{\pi}{2} \]
\[ \beta = \ln \left[ \frac{\sin \theta_i}{n} - \sqrt{\left( \frac{\sin \theta_i}{n} \right)^2 - 1} \right]. \quad (60) \]

### 14 Reflection and refraction of a plane acoustical wave

Consider two media that are separated by a plane interface and that have densities \( \rho_1 \) and \( \rho_2 \) and sound velocities \( v_1 \) and \( v_2 \). In linear acoustics the sound pressure \( p(r, t) \) is the solution of the wave equation

\[ (\nabla^2 - \frac{1}{v^2} \frac{\partial^2}{\partial t^2})p(r, t) = 0. \quad (61) \]

Therefore, a plane time-harmonic acoustical wave can be expressed as follows

\[ p(r, t) = \text{Re}\{p(r)e^{-i\omega t}\} \quad (62) \]

where

\[ p(r) = p_0 e^{ikr} \quad (63) \]
Here the amplitude $p_0$ is a constant. A plane harmonic pressure wave is incident upon a plane interface as illustrated in Fig. 8. The incident wave $p^i(r)$ is given by

$$p^i(r) = p_0^i e^{i k^i \cdot r}; \quad k^i = k_x^i \hat{e}_x + k_z^i \hat{e}_z. \quad (64)$$

The incident wave gives rise to a reflected plane wave and a transmitted plane wave of the same frequency, i.e.

$$
\begin{align*}
p^r(r, t) &= \text{Re}\{p^r(r) e^{-i \omega t}\} \\
p^r(r) &= p_0^r e^{i k^r \cdot r}; \quad k^r = k_x^r \hat{e}_x + k_y^r \hat{e}_y + k_z^r \hat{e}_z \\
p^t(r, t) &= \text{Re}\{p^t(r) e^{-i \omega t}\} \\
p^t(r) &= p_0^t e^{i k^t \cdot r}; \quad k^t = k_x^t \hat{e}_x + k_y^t \hat{e}_y + k_z^t \hat{e}_z.
\end{align*}
\quad (65)$$

The particle velocity $v$ is given by:

$$v = \text{Re}\{v^q(r) e^{-i \omega t}\} \quad (66)$$

where

$$v^q(r) = \frac{1}{i \omega \rho^q} \nabla p^q \quad ; \quad (q = i, r, t) \quad (67)$$

with $\rho^i = \rho^r = \rho_1$ and $\rho^t = \rho_2$. The boundary conditions that must be satisfied at the interface $z = 0$, are that $p$ and $\mathbf{v} \cdot \hat{e}_z$ must be continuous, i.e.

$$
\begin{align*}
\left[ p^i + p^r - p^t \right]_{z=0} &= 0 \quad (68) \\
\left[ (\mathbf{v}^i + \mathbf{v}^r - \mathbf{v}^t) \cdot \hat{e}_z \right]_{z=0} &= 0. \quad (69)
\end{align*}
$$

14.1 Snell’s law and the reflection law

Derive Snell’s law and the reflection law.

14.2 Reflection and transmission coefficients

Determine the reflection coefficient

$$R = \frac{p_0^r}{p_0^i} \quad (70)$$

and the transmission coefficient

$$T = \frac{p_0^t}{p_0^i}. \quad (71)$$
14.3 Comparison with electromagnetic waves

Compare the results with the reflection and transmission coefficients for a plane electromagnetic wave.

15 Fourier representation of a real function

Given the Fourier transform pair

\[
\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega \\
f(\omega) = \int_{-\infty}^{\infty} \hat{f}(t) e^{i\omega t} dt
\]  

(72)

show that when \( \hat{f}(t) \) is real, we can express \( \hat{f}(t) \) as follows

\[
\hat{f}(t) = 2Re \left\{ \frac{1}{2\pi} \int_{0}^{\infty} f(\omega) e^{-i\omega t} d\omega \right\}. 
\]  

(73)

(Hint: Start by showing \( f^*(\omega) = f(-\omega) \).

16 Convolution theorem, autocorrelation theorem, and Parseval’s theorem

Let \( G(k_x, k_y) \) and \( g(x, y) \), and also \( H(k_x, k_y) \) and \( h(x, y) \) be Fourier transform pairs so that

\[
A(k_x, k_y) = \int \int_{-\infty}^{\infty} a(x, y) e^{-i(k_x x + k_y y)} dx dy \\
a(x, y) = \left( \frac{1}{2\pi} \right)^2 \int \int_{-\infty}^{\infty} A(k_x, k_y) e^{i(k_x x + k_y y)} dk_x dk_y
\]  

(74)

if we let \( a \) and \( A \) stand for either \( g \) and \( G \) or \( h \) and \( H \).

16.1 Convolution

Prove the convolution theorem

\[
\left( \frac{1}{2\pi} \right)^2 \int \int_{-\infty}^{\infty} G(k_x, k_y) H(k_x, k_y) e^{i(k_x x + k_y y)} dk_x dk_y = \int \int_{-\infty}^{\infty} g(x', y') h(x-x', y-y') dx' dy'.
\]  

(75)
16.2 Autocorrelation

Prove the autocorrelation theorem

\[
\left( \frac{1}{2\pi} \right)^2 \int \int \left| G(k_x, k_y) \right|^2 e^{i(k_x x + k_y y)} dk_x dk_y = \int \int \left| g(x', y') \right|^2 g^*(x' - x, y' - y) dx' dy'.
\]

(76)

16.3 Parseval’s theorem

Prove Parseval’s theorem

\[
\left( \frac{1}{2\pi} \right)^2 \int \int \left| G(k_x, k_y) \right|^2 dk_x dk_y = \int \int \left| g(x, y) \right|^2 dx dy.
\]

(77)

17 Angular-spectrum representation of a spherical wave (Weyl’s formula)

My work has always tried to unite the true with the beautiful and when I had to choose one or the other, I usually chose the beautiful. Hermann Weyl

The field associated with a diverging spherical wave with centre at the origin is given by

\[
u(x, y, z) = \frac{e^{ikR}}{R}; \quad R = \sqrt{x^2 + y^2 + z^2}.
\]

(78)

In the plane \( z = 0 \) we have

\[
u(x, y, 0) = \frac{e^{ik\sqrt{x^2 + y^2}}}{\sqrt{x^2 + y^2}}.
\]

(79)
According to the angular-spectrum representation, we have

\[ u(x, y, z) = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(k_x, k_y) e^{i(k_x x + k_y y + k_z z)} dk_x dk_y \]  

(80)

where the angular spectrum \( U(k_x, k_y) \) is given by

\[ U(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y, 0) e^{-i(k_x x + k_y y)} dx dy \]  

(81)

and where

\[ k_z = \begin{cases} \sqrt{k^2 - k_x^2 - k_y^2} & \text{for } k^2 > k_x^2 + k_y^2 \\ i\sqrt{k_x^2 + k_y^2 - k^2} & \text{for } k^2 < k_x^2 + k_y^2. \end{cases} \]  

(82)

### 17.1 The angular spectrum

Show that the angular spectrum of a spherical wave can be written

\[ U(k_x, k_y) = 2\pi \int_0^\infty e^{ikt} J_0(q, t) dt \]  

(83)

where

\[ q^2 = k_x^2 + k_y^2. \]  

(84)

### 17.2 Weyl’s plane-wave expansion of a spherical wave

Use the formulas

\[ \int_0^\infty J_0(ax) \sin(bx) dx = \begin{cases} 0 & 0 < b < a \\ \frac{1}{\sqrt{b^2 - a^2}} & 0 < a < b \end{cases} \]  

(85)

and

\[ \int_0^\infty J_0(ax) \cos(bx) dx = \begin{cases} \frac{1}{\sqrt{a^2 - b^2}} & 0 < b < a \\ \infty & a = b \\ 0 & 0 < a < b \end{cases} \]  

(86)

Use this result to find a plane-wave expansion of a spherical wave.
18  The Airy diffraction pattern

Show that
\[
\int_0^1 J_0(vt)tdt = \frac{1}{2} \left( \frac{2J_1(v)}{v} \right). \tag{88}
\]

Hint: The following recursion formulas apply to Bessel functions:
\[
\frac{d}{dx} \left\{ x^{n+1} J_{n+1}(x) \right\} = x^{n+1} J_n(x) \tag{89}
\]
\[
\frac{d}{dx} \left\{ x^{-n} J_n(x) \right\} = -x^{-n} J_{n+1}(x). \tag{89}
\]

19  Integrated energy of the Airy diffraction pattern

19.1  \( \frac{J_1^2(x)}{x} \)

Use the recursion formulas in exercise 18 with \( n = 0 \) to show that
\[
\frac{J_1^2(x)}{x} = -\frac{1}{2} \frac{d}{dx} \left[ J_0^2(x) + J_1^2(x) \right]. \tag{90}
\]

19.2  Encircled energy

Fraunhofer diffraction through a circular aperture gives rise to the Airy diffraction pattern, so that the intensity is given by
\[
I(v) = C \left[ \frac{2J_1(v)}{v} \right]^2, \tag{91}
\]
where \( C \) is a constant and where \( v \) is a dimensionless co-ordinate given by
\[
v = k \frac{a}{z_2} r. \tag{92}
\]
Here \( r \) is the distance from the optical axis \( z = 0 \) to the observation point. The integrated energy \( E(v_0) \) inside a circle with dimensionsless radius \( v_0 \) is given by the integral of \( I(v) \) over the circular area, i.e.
\[
E(v_0) = \int_0^{v_0} \int_0^{2\pi} I(v)v dv d\psi = 2\pi \int_0^{v_0} I(v)v dv. \tag{93}
\]
The ratio of \( E(v_0) \) to the total energy is called the encircled energy, and is given by
\[
L(v_0) = \frac{E(v_0)}{E(\infty)} = \frac{\int_0^{v_0} \frac{J_1^2(v)}{v} dv}{\int_0^{\infty} \frac{J_1^2(v)}{v} dv}. \tag{94}
\]
Use the result from 19.1 and the limiting values:

\[
\begin{align*}
J_0(0) &= 1 \\
J_1(0) &= J_1(\infty) = J_0(\infty) = 0
\end{align*}
\]  \hspace{1cm} (95)

to show that:

\[
L(v_0) = 1 - J_0^2(v_0) - J_1^2(v_0).
\]  \hspace{1cm} (96)

20 Percentage energy inside the first, second, and third dark rings of the Airy diffraction pattern

20.1 Computer program

Make a computer program for determining \( J_0(x) \) and \( J_1(x) \) from the approximations in section 9.4 in Abramowitz and Stegun [1]. The approximation for \( J_0 \) for \( x \) between 0 and 3 is given by:

\[
J_0 = 1 - 2.2499997(x/3)^2 + 1.2656208(x/3)^4 - 0.3163866(x/3)^6
+ 0.0444479(x/3)^8 - 0.0039444(x/3)^{10} + 0.0002100(x/3)^{12}
\]  \hspace{1cm} (97)

and for \( x \) between 3 and \( \infty \), we have

\[
J_0 = x^{-1/2} \cos(\theta_0)f_0
\]  \hspace{1cm} (98)

where

\[
f_0 = 0.79788456 - 0.00000077(3/x) - 0.00552740(3/x)^2 - 0.00009512(3/x)^3
+ 0.00137237(3/x)^4 - 0.00072805(3/x)^5 + 0.00014476(3/x)^6
\]  \hspace{1cm} (99)

\[
\theta_0 = x - 0.78539816 - 0.04166397(3/x) - 0.00003954(3/x)^2 - 0.00262573(3/x)^3
- 0.0054125(3/x)^4 - 0.00029333(3/x)^5 + 0.00013558(3/x)^6.
\]  \hspace{1cm} (100)

For \( x \) between 0 and 3, \( J_1 \) is given by

\[
J_1 = (1/2) - 0.56249985(x/3)^2 + 0.21093573(x/3)^4 - 0.03954289(x/3)^6
+ 0.00443319(x/3)^8 - 0.00031761(x/3)^{10} + 0.00001109(x/3)^{12}x
\]  \hspace{1cm} (101)
and for \( x \) between 3 and \( \infty \), we have

\[
J_1 = x^{-1/2} \cos(\theta_1) f_1
\]  

(102)

where

\[
f_1 = 0.79788456 + 0.00000156(3/x) + 0.01659667(3/x)^2 + 0.00017105(3/x)^3
- 0.00249511(3/x)^4 + 0.00113653(3/x)^5 - 0.00020033(3/x)^6
\]  

(103)

\[
\theta_1 = x - 2.35619449 + 0.12499612(3/x) + 0.00005650(3/x) - 0.00637879(3/x)^3
+ 0.00074348(3/x)^4 + 0.00079824(3/x)^5 - 0.00029166(3/x)^6.
\]  

(104)

## 20.2 Percentage energy

The first four zeros \( J_1(x) \) are at \( x = 0 \), \( x = 3.833 \), \( x = 7.016 \), and \( x = 10.174 \). Use the computer program from 20.1 and the result from exercise 19.2 to determine the percentage energy inside the first, second, and third dark rings of the Airy diffraction pattern.

## 21 Fresnel diffraction through an infinitely large circular aperture

From equation 11.2.7 in the lecture notes [5] we have for the field diffracted through a circular aperture:

\[
u_I = C \int_0^1 J_0(vt)e^{i\frac{1}{2}ut^2}tdt
\]  

(105)

where

\[
v = \frac{ka}{z_2}r
\]

\[
u = \frac{ka}{z_2}t
\]

\[
C = \frac{2\pi a^2}{i\lambda z_2}e^{i\phi}
\]

\[
\phi = k(z_2 + \frac{r^2}{2z_2}).
\]  

(106)

Here \( a \) is the aperture radius, and the incident field is a normally incident plane wave, as illustrated in Fig. 10.
21.1 Change of variable: $vt \rightarrow x$

Show that $u_I$ can be expressed as:

$$u_I = \frac{z_2 e^{ik\phi}}{kr^2} \int_0^v J_0(x)e^{iBx^2}x dx,$$

where

$$B = \frac{z_2}{2kr^2}.$$  \hspace{1cm} (107)

21.2 Infinitely large aperture

When the aperture radius becomes infinitely large, we get $v = k \alpha z \rightarrow \infty$, so that

$$u_I = \frac{z_2 e^{ik\phi}}{kr^2} \int_0^\infty J_0(x)e^{iBx^2}x dx.$$  \hspace{1cm} (109)

Use integration by parts together with the results

$$\int_0^\infty \sin(ax^2)J_1(bx)dx = \frac{1}{b} \sin \left( \frac{b^2}{4a} \right)$$

$$\int_0^\infty \cos(ax^2)J_1(bx)dx = \frac{2}{b} \sin^2 \left( \frac{b^2}{8a} \right)$$  \hspace{1cm} (110)

to show that (109) can be written

$$u_I = e^{ikz_2}.$$  \hspace{1cm} (111)

What is the physical interpretation of this result?

22 Diffraction by a half-plane

The diffracted field resulting when a plane wave is normally incident upon the edge of a half-plane, is given in terms of so-called detour parameters. The detour parameter $\xi$ associated with the incident wave is defined such that $(\xi)^2$ is the difference between the phase measured along the diffracted
ray and along the incident ray. Also, $\xi^i$ is defined such that $\xi^i > 0$ in the shadow zone of the incident wave, whereas $\xi^i < 0$ in the lit area. From this definition and Fig. 11, we have

$$ (\xi^i)^2 = k(s - D) \quad (112) $$

$$ \text{sgn}(\xi^i) = \text{sgn}(\theta_0 - \theta). \quad (113) $$

22.1 Detour parameter associated with the incident wave

Show from (112) and (113) that

$$ \xi^i = -\sqrt{2ks} \sin \frac{1}{2}(\theta - \theta_0). \quad (114) $$

22.2 Detour parameter associated with the reflected wave

The detour parameter $\xi^r$ associated with the reflected wave is defined such that $(\xi^r)^2$ is the difference between the phase measured along the diffracted ray and along the reflected ray. Also, $\xi^r$ is defined such that $\xi^r > 0$ in the shadow zone of the reflected wave, whereas $\xi^r < 0$ in the area that is lit by the reflected wave. Draw a figure and show that

$$ \xi^r = \sqrt{2ks} \sin \frac{1}{2}(\theta + \theta_0). \quad (115) $$
23 Diffraction through a circular aperture - axial intensity

When a plane wave is normally incident upon a circular aperture, the intensity in the diffraction pattern is given by

\[ I = \left( \frac{\pi a^2}{\lambda z^2} \right)^2 2 \int_0^1 J_0(vt) e^{i\frac{1}{2}vt^2} t dt \right)^2. \]  \hspace{1cm} (116)

Show that along the axis \( v = 0 \) the intensity becomes

\[ I = \left( \frac{\pi a^2}{\lambda z^2} \right)^2 \left( \sin \left( \frac{u}{4} \right) \right)^2. \] \hspace{1cm} (117)

24 Poisson’s spot

In the diffraction problem illustrated in Fig. 12 a point source in the half-space \( z < 0 \) radiates a field \( u^i \) which is incident upon an aperture \( A \) in the plane \( z = 0 \). When \( kR_2 >> 1 \), the diffracted field is given by Rayleigh-Sommerfeld’s first diffraction integral, i.e.

\[ u_I = \frac{1}{i\lambda} \int_A u^i(x, y, 0) e^{ikR_2} \frac{z_2}{R_2} dxdy \] \hspace{1cm} (118)

where

\[ R_2 = \sqrt{(x-x_2)^2 + (y-y_2)^2 + z_2^2}. \] \hspace{1cm} (119)

For a normally incident plane wave \( u^i(x, y, 0) = 1 \), and we get

\[ u_I = \frac{1}{i\lambda} \int_A \frac{e^{ikR_2}}{R_2} \frac{z_2}{R_2} dxdy. \] \hspace{1cm} (120)
24.1 Diffraction of a spherical wave through a circular aperture

If \( u \) is due to a point source at \((x_1, y_1, -z_1)\), we get

\[
u_I = \frac{1}{i\lambda} \int \int_A \frac{e^{i(kR_1+R_2)}}{R_1R_2} \frac{z_2}{R_2} dxdy \tag{121}
\]

where

\[
R_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2 + z_1^2}.
\tag{122}
\]

24.1 Diffraction of a spherical wave through a circular aperture

In the lectures we have shown that when a normally incident plane wave is diffracted through a circular aperture, the first Rayleigh-Sommerfeld diffraction integral in (120) can be expressed as

\[
u_{IA} = -iC_1 \frac{\pi a^2}{\lambda z_2} 2 \int_0^1 J_0(vt)e^{i\frac{1}{2}u^2} t dt \tag{123}
\]

where

\[
\begin{align*}
v &= \frac{k a}{k z_2} \\
u &= \frac{k a^2}{z_2} \\
C &= e^{ik(z_2 + x_2^2 + y_2^2)}/(v^2).
\end{align*}
\tag{124}
\]

Equation (123) applies provided that the paraxial approximation and the Fresnel approximation are satisfied. Let \( x_1 = y_1 = 0 \) and use the same approximations to show that (121) can be expressed as

\[
u_{IA} = -iC_2 \frac{\pi a^2}{\lambda z_1 z_2} 2 \int_0^1 J_0(vt)e^{i\frac{1}{2}\bar{u}^2} t dt \tag{125}
\]

where

\[
C_2 = C_1 e^{ikz_1} ; \quad \bar{u} = ka^2 \left( \frac{1}{z_1} + \frac{1}{z_2} \right). \tag{126}
\]

24.2 Opaque disc

Let the incident field be a normally incident plane wave, and let the circular aperture be replaced by a circular disc. Show that the diffracted field then is given by

\[
u_{IS} = e^{ikz_2} - \nu_{IA} \tag{127}
\]

where \( \nu_{IA} \) is given by (123).
25 Axial field - incident plane wave

Show that on the axis \( v = 0 \) we get from (127)

\[
u_L S = e^{ik\left(z_2 + \frac{x^2}{z_2}\right)}
\]

which implies that the intensity everywhere on the axis behind an opaque circular disc is the same as the intensity of the incident plane wave! This bright spot is Poisson’s spot, named after Poisson who first predicted it on the basis of Fresnel’s wave theory. Poisson used this bright spot as a proof against the wave theory. But when Arago soon afterwards carried out the experiment, he observed the bright spot on the axis.

25.1 Axial field - incident spherical wave

Repeat the derivations in exercises 24.2 and 25 for an incident diverging spherical wave, so that (125) applies. Consider the special case in which \( z_1 = z_2 \) and try to simplify the result as much as possible.

26 Fraunhofer diffraction at oblique incidence and interference between the fields diffracted through two apertures

In the lectures we have shown that the diffracted field in the Fraunhofer zone is given by

\[
u_I = \frac{C_1}{i\lambda z_2} \int \int_{\infty} u^i(x, y, 0)t(x, y)e^{-i(k_0^x x + k_0^y y)}dxdy
\]

(129)

where

\[
C_1 = e^{ik\left(z_2 + \frac{x^2 + y^2}{z_2}\right)} ; \quad k_0^x = \frac{k x_2}{z_2} ; \quad k_0^y = \frac{k y_2}{z_2}
\]

(130)

and where \( u^i \) is the incident field and \( t(x, y) \) is the transmission function of the aperture. Let us assume that \( t(x, y) \) has the value 1 when \( (x, y) \) lies inside the aperture \( A \) and the value zero when \( (x, y) \) lies outside \( A \).

26.1 Fourier representation at oblique incidence

Show that for an obliquely incident plane wave with wave vector

\[
k^i = k_x^i \hat{e}_x + k_y^i \hat{e}_y + k_z^i \hat{e}_z
\]

(131)
the diffracted field becomes

\[ u_I = \frac{C_1}{i\lambda z_2} \int \int_A e^{-i(K_x x + K_y y)} dxdy \]  (132)

where

\[ K_x = k_0^x - k^i_x \quad ; \quad K_y = k_0^y - k^i_y. \]  (133)

26.2 Airy diffraction pattern at oblique incidence

Let \( k^i \) lie in the \((y, z)\) plane and make an angle \( \theta_0 \) with the positive z axis, and let the aperture be circular with radius \( a \) and with centre at \( x = y = 0 \), as shown in Fig. 13. Determine the diffracted field in the Fraunhofer zone. Simplify the expression as much as possible, and compare the result with that found previously for Fraunhofer diffraction through a circular aperture when the plane wave is normally incident.

26.3 Aperture displacement

Show that if we displace the aperture \( A \) in the plane \( z = 0 \) in equation (132) a distance \( s \) in a direction that makes an angle \( \beta_0 \) with the positive x axis, then the diffracted field \( u'_I \) due to the new aperture \( A' \) is given by

\[ u'_I = u_I e^{-i(K_x \xi_1 + K_y \eta_1)} \]  (134)

where

\[ \xi_1 = s \cos \beta_0 \quad ; \quad \eta_1 = s \sin \beta_0 \]  (135)

is the origin in a displaced co-ordinate system \((x', y')\) that is related to \( A' \) in the same way as the original co-ordinate system \((x, y)\) is related to \( A \).
26.4 Interference

Use the result from exercise 26.3 to show that the diffracted intensity due to both $A$ and $A'$ is given by

$$I = 4I_0 \cos^2 \left(\frac{1}{\lambda} \delta \right)$$  \hspace{1cm} (136)

where $I_0$ is the intensity due to $A$ or $A'$ alone, and where

$$\delta = K_x \xi_1 + K_y \eta_1.$$  \hspace{1cm} (137)

26.5 Example

Sketch the diffraction pattern given by (136) for the case in which we have two circular apertures with radius $a = 0.1$ mm and normal incidence. Let the other parameters be $\beta_0 = 0$, $s = 1$ mm, $\lambda = 0.5$ $\mu$m, and $z_2 = 10$ m. How large is the diameter of the Airy disc, and how many dark interference fringes are there inside the Airy disc?

27 Dipole radiation

Let

$$\phi(r) = \frac{e^{ikr}}{r}$$  \hspace{1cm} (138)

where

$$r = \sqrt{x^2 + y^2 + z^2}$$  \hspace{1cm} (139)

and let $\hat{n}$ be a unit vector that does not vary with $x$, $y$, and $z$.

27.1 1st Step

Use the relation

$$\nabla \times (a\phi) = \phi(\nabla \times a) - a \times \nabla \phi$$  \hspace{1cm} (140)

which applies to general functions $a$ and $\phi$ to show that

$$\nabla \times (\hat{n}\phi) = \frac{\partial \phi}{\partial r} \hat{e}_r \times \hat{n}$$  \hspace{1cm} (141)

where

$$\hat{e}_r = \frac{r}{|r|} = \frac{x}{r} \hat{e}_x + \frac{y}{r} \hat{e}_y + \frac{z}{r} \hat{e}_z.$$  \hspace{1cm} (142)
27.2 2nd Step

Show that
\[ \nabla \times \nabla \times (\hat{n} \phi) = \frac{\partial \phi}{\partial r} \nabla \times (\hat{e}_r \times \hat{n}) + \hat{e}_r \times (\hat{e}_r \times \hat{n}) \frac{\partial^2 \phi}{\partial^2 r}. \] (143)

27.3 3rd Step

Use the relation
\[ \nabla \times (a \times b) = (b \cdot \nabla) a - (a \cdot \nabla) b + a(\nabla \cdot b) - b(\nabla \cdot a) \] (144)
which applies to general vector functions \( a \) and \( b \), to show that
\[ \nabla \times (\hat{e}_r \times \hat{n}) = (\hat{n} \cdot \nabla) \hat{e}_r - \hat{n}(\nabla \cdot \hat{e}_r). \] (145)

27.4 4th Step

Show that
\[ \nabla \cdot \hat{e}_r = \frac{2}{r}. \] (146)

27.5 5th Step

Show that
\[ (\hat{n} \cdot \nabla) \hat{e}_r = \frac{1}{r} [\hat{n} - \hat{e}_r (\hat{n} \cdot \hat{e}_r)]. \] (147)

27.6 6th Step

Show that
\[ \nabla \times \nabla \times (\hat{n} \phi) = \hat{e}_r (\hat{e}_r \cdot \hat{n}) \left[ \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} \right] - \hat{n} \left[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right]. \] (148)

27.7 7th Step

Show that
\[ \frac{\partial \phi}{\partial r} = ik \left( 1 + \frac{i}{kr} \right) \phi \] (149)
and
\[ \frac{\partial^2 \phi}{\partial r^2} = \left[ \frac{2}{r^2} - k^2 \left( 1 + \frac{2i}{kr} \right) \right] \phi. \] (150)
28 Field radiated by an \( x \) polarised dipole

In the lecture notes [5] it is shown that the electromagnetic field radiated by a dipole that is placed at the origin and polarised in the direction of the unit vector \( \hat{n} \), is given by

\[
B = \frac{\mu I}{c} \nabla \times \left( \hat{n} \frac{e^{ikr}}{r} \right) \quad (151)
\]

and

\[
E = \frac{i\mu I}{ck} \nabla \times \nabla \times \left( \hat{n} \frac{e^{ikr}}{r} \right) \quad (152)
\]

where \( r = \sqrt{x^2 + y^2 + z^2} \) is the distance from the dipole to the observation point. Use the results from exercise 27 to show that when \( \hat{n} = \hat{e}_x \), equations (151) and (152) can be written as follows

\[
B_x = 0 \quad (153)
\]

\[
B_y = \frac{\mu I}{c} \frac{z}{r} ik \frac{e^{ikr}}{r} P' \quad (154)
\]

\[
B_z = -\frac{\mu I}{c} \frac{y}{r} ik \frac{e^{ikr}}{r} P' \quad (155)
\]

\[
E_x = \frac{i\mu I}{ck} \frac{e^{ikr}}{r} \left\{ 1 + \frac{i}{kr} P' - \left( \frac{x}{r} \right)^2 Q' \right\} \quad (156)
\]

\[
E_y = -\frac{i\mu I}{ck} \frac{x}{r} \frac{y}{r} Q' \quad (157)
\]

\[
E_z = -\frac{i\mu I}{ck} \frac{x}{r} \frac{z}{r} Q' \quad (158)
\]

where \( Q' \) and \( P' \) are defined as follows

\[
Q' = 1 + \frac{3i}{kr} - \frac{3}{(kr)^2} \quad (159)
\]

\[
P' = 1 + \frac{i}{kr}.
\]

29 Edge diffraction: Uniform asymptotic expansion

In the lecture notes [5] it is shown how one can deal with integrals of the form

\[
J = \int_{x_1}^{x_2} g(x) e^{ikf(x)} dx. \quad (160)
\]

According to section 14.1 in the lecture notes [5], the solution of a two-dimensional diffraction problem can be expressed in terms of an integral of
29.1 Integral expression

the form given above. One can evaluate this integral asymptotically by first
determining the stationary point of the phase function \( f(x) \). The integrand
oscillates very rapidly in the whole integration interval except in a small neigh-
bourhood around each stationary point of \( f(x) \). In the lecture notes [5] \( f(x) \)
is expanded around a stationary point \( x_s \), so that the first-order term in the
Taylor series for \( f(x) \) vanishes. However, this method becomes invalid when
the the stationary point lies close to one of the end points \( x_1 \) or \( x_2 \). If we
start with the integral

\[
J = \int_{x_1}^{\infty} g(x)e^{ikf(x)} dx
\]  

(161)

and expand \( f(x) \) around the lower end point \( x_1 \), we avoid problems when \( x_s \)
approaches \( x_1 \). Then we obtain the following result

\[
J_1 = \frac{1}{|f_2|} \sqrt{\frac{\pi}{k}} \left\{ g_0 e^{i(kf_0 - \epsilon v^2)} \sqrt{\frac{\pi}{2}} \left[ C(\infty) + i\epsilon S(\infty) - [C(v) + i\epsilon S(v)] \right] \right\}
\]  

(162)

where \( f_i = f^{(i)}(x_1) \), \( g_0 = g(x_1) \), \( \epsilon = \text{sgn}(f_2) \), and \( v = |\frac{f_1}{f_2}|\sqrt{k|f_2|} \). Here \( S \) and \( C \) are the Fresnel integrals defined in equations (11.3.12) in the lecture
notes [5].

29.1 Integral expression

Show that (162) can be expressed as

\[
J_1 = \frac{1}{|f_2|} \sqrt{\frac{\pi}{k}} g_0 e^{i(kf_0 - \epsilon v^2)} \int_v^\infty e^{i\epsilon s^2} ds.
\]  

(163)

29.2 Modified integral expression

Show that \( J_1 \) can be expressed as follows

\[
J_1 = \frac{1}{2} \sqrt{\frac{\pi}{|f_2|}} g_0 e^{i(kf_0 + \epsilon(\frac{\pi}{4} - v^2))} \text{erfc}(ve^{-\left(\frac{\pi}{4}\right)})
\]  

(164)

where the complementary error function is defined by

\[
\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-s^2} ds
\]  

(165)

Hint: Use Cauchy-Goursat’s theorem, explained e.g. in Churchill and Brown [3],
and use the substitution \( t = se^{-\frac{i\pi}{4}} \) when \( \epsilon = +1 \) and \( t = se^{\frac{i\pi}{4}} \) when \( \epsilon = -1 \).
29.3 Stationary point near end point

Determine $J_1$ when $x_s \to x_1$ (i.e. when $v \to 0$). Compare this asymptotic contribution with that obtained from an interior stationary point. The asymptotic contribution from an interior stationary point is given in equation (14.2.13) in the lecture notes [5] as

$$J_S = \sqrt{\frac{\pi}{k|f_2|}} g_0 e^{i(kf_0 + \frac{\pi}{4} \text{sgn}(f_2))}. \quad (166)$$

29.4 Asymptotic expansion

In Abramowitz and Stegun [1] the following asymptotic formula for the complementary error function is given by

$$\sqrt{\pi e^{-z^2}} \text{erfc}(z) \sim 1 + \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdots (2m-1)}{(2z^2)^m}, \quad (167)$$

when $|z| \to \infty$ and $|\arg(z)| < \frac{3\pi}{4}$. Use the first term in this series to show that when $|z| \to \infty$, we get

$$\text{erfc}[ve^{-\pi \frac{z}{4}}] \sim \frac{e^{i(\epsilon v^2 + \epsilon \frac{\pi}{4})}}{\sqrt{\pi \epsilon}}. \quad (168)$$

29.5 Stationary point far from end point

Determine $J_1$ when $x_s \ll x_1$ (i.e. when $v \to \infty$). According to equation (8.14a) in [4], the diffracted edge contribution can be expressed as

$$J_E \sim \left( \left. \frac{e^{ikf(x)}}{ikf'(x)} \sum_{n} L_n(x) \right|_{x_1}^{x_2} \right), \quad (169)$$

where $L_0 = g_0$. Here we have assumed that there are no stationary points between $x_1$ and $x_2$. If $x_s \ll x_1$, it suffices to use the first term in the asymptotic expansion. If we consider diffraction by a half-plane, so that $x_2 \to \infty$, we obtain

$$J_E \sim \left. \left( \frac{e^{ikf(x)}}{ikf'(x)} g_0 \right) \right|_{x_1}^{\infty} = \frac{g_0}{ikf_1} e^{ikf_0}. \quad (170)$$

Let $x_s \ll x_1$ (so that $v \to \infty$) in the uniform asymptotic result in (164) and compare the result thus obtained with the non-uniform result in (170).
References


